Lectures on Symplectic Geometry

reference: by Cannes da Silva. 2017 Fall

§ Introduction.

Def. $\omega \in \Omega^2(M)$ Symplectic

if (1) non-degen. $T_xM \xrightarrow[\cong]{\lrcorner\omega} T_x^*M$ [0th order]

(2) integrable $d\omega = 0$ [1st order]

Compare w/ Complex mfd:

(1) $J_x : T_xM \longrightarrow T_xM$, $J_x^2 = -\mathrm{id}$ $\left(\Rightarrow (T_xM, J_x) \cong (\mathbb{C}^n, J_{std})\right)$

(2) $\mathcal{N} = 0$ $\left(\Rightarrow (M, J) \overset{loc.}{\cong} (\mathbb{C}^n, J_{std})\right)$

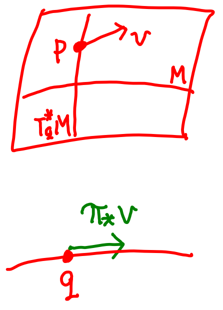
Eg. $M = T^*X$, $\omega_{can} = -d\alpha = dq \wedge dp$ $(\alpha = p\,dq)$

(i.e. exact sympl. str.)

$\alpha \in \Omega^1(M)$

i.e. ∀ $(q, p) \in M$ & $v \in T_{(q,p)}M$

$(p \in T_q^*M)$



$$\begin{tikzcd} T^*X = M \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$\rightsquigarrow \alpha_{(q,p)}(v) = \underset{T_q^*X.}{\underset{\cap}{p}} \left(\underset{T_qX.}{\underset{\cap}{\pi_* v}}\right)$

In loc. coord. $X \ni (q^1, \dots, q^n) \rightsquigarrow p_1 dq^1 + \dots + p_n dq^n \in T_q^*X$

$\rightsquigarrow$ loc. coord. $M \ni (q^1, \dots, q^n, p_1, \dots, p_n)$

$\Rightarrow \quad \alpha = \sum_{j=1}^{n} p_j \, dq^j.$ (indep. of coord.).

$\omega = \sum dq^j \wedge dp_j.$

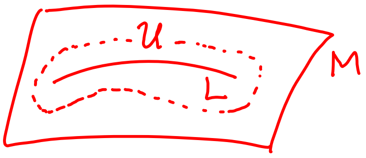
- $\mathrm{Diff}(X) \hookrightarrow \mathrm{Sympl}(T^*X, \omega).$

(1) ω non-degen. $\overset{\text{Linear}}{\underset{\text{alg.}}{\Longleftrightarrow}}$ $(T_xM, \omega_x) \cong (T^*\mathbb{R}^n, \omega_{can})$

(2) $d\omega = 0$ $\Longleftrightarrow$ $(M, \omega) \overset{loc.}{\cong} (T^*\mathbb{R}^n, \omega_{can})$.

Darboux's thm. (prove later)

Def. $L^n \subset (M^{2n}, \omega)$ Lagrangian



$\overset{\Delta}{\Longleftrightarrow}$ $\omega|_L = 0$

$\overset{\text{Darboux}}{\Longleftrightarrow}$ $(L \subset \underset{\swarrow\text{open nbd. in } M}{U}, \omega) \cong (L \subset T^*L, \omega_{can})$

(II) Linear Symplectic Geometry

$GL(V) \curvearrowright V^* \otimes V^* \simeq \underset{g}{Sym^2 V^*} + \underset{\omega}{\Lambda^2 V^*}$ for $V \simeq \mathbb{R}^m$

- $\exists$ basis s.t. $g(v,v) = v^T \begin{pmatrix} I_p & & \\ & -I_q & \\ & & O_r \end{pmatrix} v$

  $GL(V)$-orbits $\longleftrightarrow$ $(p,q,r) \in \mathbb{N}^3$ w/ $p+q+r = m$

  non-degen. $\Longleftrightarrow$ $r = 0$

  pos. definite $\Longleftrightarrow$ $p = m$ $(q = r = 0)$

  $\{\text{pos. def. } g \text{ on } V\} \simeq GL(m, \mathbb{R})/O(m)$

- $\exists$ basis s.t. $\omega(v,v) = v^T \begin{pmatrix} O & & & \\ & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \end{pmatrix} v$

  $GL(V)$-orbits $\longleftrightarrow$ $n = 0, 1, \dots, [\frac{m}{2}]$

  $\{\text{sympl. } \underset{m=2n}{\omega} \text{ on } V\} \simeq GL(2n, \mathbb{R})/Sp(2n, \mathbb{R})$

$Sp(2n, \mathbb{R}) = Aut(\mathbb{R}^{2n}, \omega_{std})$, $\omega_{std} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

- $\underline{sp}(2n, \mathbb{R}) \overset{\text{Ex.}}{=} \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : B = B^T, C = C^T \right\}$, $\dim = n(2n+1)$

- ω nondegen. $\overset{\text{Ex.}}{\Longleftrightarrow}$ $\omega^n \neq 0 \in \Lambda^{2n} V^* \approx \mathbb{R}$.

$\rightsquigarrow$ $Sp(2n, \mathbb{R}) \leqslant SL(2n, \mathbb{R})$.

§ Linear Subspaces

$$V \simeq \mathbb{R}^m \qquad \mathrm{Aut}(V) \simeq GL(m, \mathbb{R})$$

$$Gr(r, V) := \{ r \text{ dim. linear subsp. of } V \}$$

$$\mathrm{Aut}(V) \curvearrowright \underset{\underset{S_0}{\Psi}}{Gr(r,V)} \quad \text{transitive} \quad \left(\begin{array}{l} \text{i.e. basis of } \mathbb{R}^r \\ \text{can be extended to } \mathbb{R}^m \end{array} \right)$$

$$\rightsquigarrow \qquad Gr(r,V) \cong \frac{\mathrm{Aut}(V)}{\mathrm{Aut}(V)_{S_0}} = \frac{GL(m)}{\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}}$$

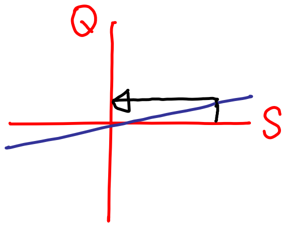
$$\cong \frac{O(m)}{O(r) O(m-r)} \quad \left(\text{if fix a metric on } V \right)$$

Tautological bundle

$$\begin{tikzcd} \mathbb{R}^r \simeq S \arrow[d] & \subset & \mathcal{S} \arrow[d] \\ {[S]} & \in & Gr(r,V) \end{tikzcd}$$

$$0 \to \mathcal{S} \to \underset{\text{(trivial bdl)}}{\underline{\mathbb{R}^m}} \longrightarrow Q \to 0 \qquad / Gr(r,V)$$

$$TGr = \mathrm{Hom}(\mathcal{S}, Q)$$
$$= \mathcal{S}^* \otimes Q$$



Similar for $\underline{\text{complex}}$ case:

$$(V, J) \cong (\mathbb{C}^n, \cdot i) \qquad m = 2n$$

$$\rightsquigarrow \quad Gr_{\mathbb{C}}(k, n) \subset Gr(2k, 2n)$$

$$JS = S$$

i.e. involution $\sigma_J : Gr(2k, 2n) \circlearrowleft, \ \sigma_J(S) = JS$

then $Gr_{\mathbb{C}} = (Gr)^{\sigma_J}$.

Symplectic case $(V, \omega) \simeq (\mathbb{R}^{2n}, \sum_{j=1}^{n} dx^j \wedge dy_j)$

$$S \leq V \rightsquigarrow S^{\perp\omega} := \{v : \omega(v, S) = 0\} \leq V$$

- $\dim S + \dim S^{\perp\omega} = \dim V$
- $(S^{\perp\omega})^{\perp\omega} = S$

i.e. involution: $\sigma_\omega : \coprod_r Gr(r, 2n) \to \coprod_r Gr(r, 2n)$

$$\sigma_\omega(S) := S^{\perp\omega} \quad (\sigma_\omega^2 = id)$$

What is $\left(\coprod_r Gr(r, 2n)\right)^{\sigma_\omega} =: LagGr(V, \omega)$?

$$S^{\perp\omega} = S \iff \begin{cases} \dim S = \frac{1}{2} \dim V \\ \omega|_S = 0 \end{cases}$$

Lagrangian subspace.

On (V, ω), $J : V \circlearrowleft$, $J^2 = -id$

called compatible

$\iff (V, J, g, \omega) \simeq (\mathbb{C}^n, i, \langle\ \rangle_{std}, \omega_{std})$

where $g(Ju, v) = \omega(u, v)$

$\overset{\Delta}{\iff}$ $g > 0$ & Hermitian $(g(Ju, Jv) = g(u, v))$

(tame if only $g > 0$)

Ex: Given compat. J on (V, ω)

$S \leq V$ Lagr. $\iff V = S \overset{\perp}{\oplus} JS$

i.e. S is a real str. on (V, J).

Ex. $S \leqslant (V, \omega)$

$\omega|_S = 0$ (called isotropic)

$\iff S \subset S^{\perp_\omega}$

$\implies V \simeq \mathbb{R}^{2n}$ (i.e. std. form) $\quad \underbrace{x^1 \cdots x^r}_{S} \cdots x^n \; y_1 \cdots y_r \cdots y_n$

$\implies \dim S \leqslant \frac{1}{2} \dim V$

So, Lagr. are biggest subsp. where $\omega|_S = 0$.

If $\dim S = n+1 \Rightarrow \omega|_S \neq 0$

the best possible is $\omega^2|_S = 0$

$\iff \overbrace{x^1 \cdots \underbrace{x^r \cdots x^n}_{S^{\perp_\omega}} \; y_1}^{S} \cdots y_r \cdots y_n$

$\iff S^{\perp_\omega} \subset S$ (called coisotropic)

Namely, $S \leqslant (V, \omega)$ coisotropic

$\overset{\triangle}{\iff} S^{\perp_\omega} \subset S$ ($\iff S^{\perp_\omega}$ isotropic) $\Rightarrow \dim S = n + r \geqslant n$

$\iff \omega^{r+1}|_S = 0$

Note $\dim S = n + r \Rightarrow \omega^r|_S \neq 0$

Namely, coisotropic (or isotropic) are subspaces for which $\omega|_S$ is as degenerate as possible.

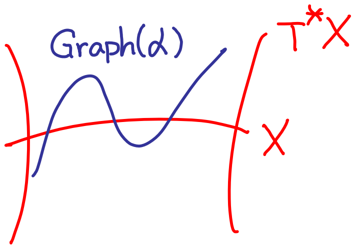
Examples of Lagrangian submfds in T^*X, ω_{can}

(0) $X \ , \ T_x^*X \underset{\text{Lagr.}}{\subset} T^*X$

0-section, fibers.

(1) $\alpha \in \Omega^1(X)$

$\rightsquigarrow \mathrm{Graph}(\alpha) \subset T^*X, \omega_{can}$



- $\mathrm{Graph}(\alpha)$ Lagr. $\iff d\alpha = 0 \in \Omega^2(X)$

Pf: $\alpha = \sum \alpha_j(x)\, dx^j$

$\Rightarrow \mathrm{Graph}(\alpha) = \{ p_j = \alpha_j(x) \ , \ j=1,\dots,n \}$

$\subseteq T^*X \qquad \omega_{can} = \sum dp_j \wedge dx^j$

$$\omega_{can}|_{\mathrm{Graph}(\alpha)} = \sum_j d \underbrace{p_j}_{\alpha_j(x)} \wedge dx^j = \sum_j \Big(\sum_k \frac{\partial \alpha_j}{\partial x^k} dx^k \Big) \wedge dx^j$$

$$= \sum_{k<j} \Big(\frac{\partial \alpha_j}{\partial x^k} - \frac{\partial \alpha_k}{\partial x^j} \Big) dx^k \wedge dx^j = d\alpha$$

(2) Conormal bundles.

submfd. $S \subset X$

$\rightsquigarrow \quad 0 \to TS \to TX|_S \to N_{S/X} \to 0$

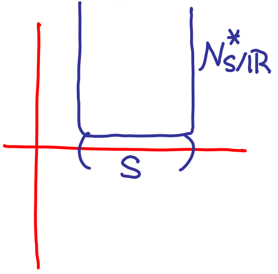
$$\begin{tikzcd} \rightsquigarrow \quad 0 \arrow[r] & N^*_{S/X} \arrow[r] \arrow[dr, hook, "\text{(Ex.) Lagr.}"'] & T^*X|_S \arrow[r] \arrow[d, phantom, "\cap"] & T^*S \arrow[r] & 0 \\ & & T^*X & & \end{tikzcd}$$

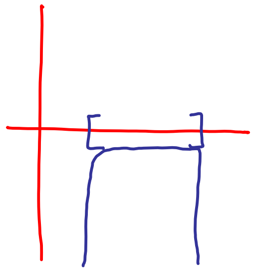
eg. $S = X \Rightarrow N^*_{S/X} = X \overset{\text{0-section}}{\subset} T^*X$

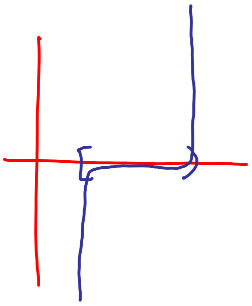
eg. $S = \{x\} \Rightarrow N^*_{S/X} = T^*_x X \underset{\text{fiber}}{\subset} T^*X$

We could take S to be open subsets,

e.g. $T^*\mathbb{R}$







- Combining (1) & (2).

$$\begin{cases} S \subset X & \text{closed submfd.} \\ \alpha \in \Omega^1(S) \text{ w/ } d\alpha = 0 \end{cases}$$

$$\Rightarrow \quad N^*_{S/X} + \alpha \underset{\text{Lagr.}}{\subset} T^*X$$

- $$f : (M_1^{2n}, \omega_1) \longrightarrow (M_2^{2n}, \omega_2)$$

$$\rightsquigarrow \text{Graph}(f) \subseteq M_1 \times \overline{M}_2 \quad \text{w/ } \omega_1 - \omega_2$$

$$f^*\omega_2 = \omega_1 \quad \Longleftrightarrow \quad \text{Graph}(f) \overset{\text{Lagr.}}{\subset} M_1 \times \overline{M}_2$$

(eg Symplectomorphisms)

- Lagrangian correspondence.

$$\begin{tikzcd} K \overset{\text{Lagr.}}{\subset} (M_1, \omega_1) \times (M_2, \omega_2) \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_1 & M_2 \end{tikzcd}$$

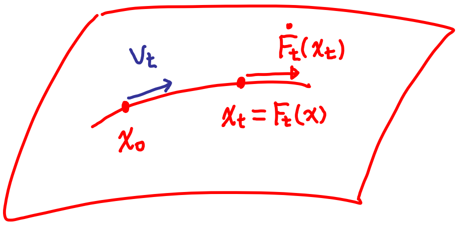
$$L \overset{\text{Lagr}}{\subset} M_1 \xRightarrow{(\text{assume } \pitchfork)} \pi_2(\pi_1^*(L) \cap K) \overset{\text{Lagr.}}{\subset} M_2$$

($\sim$ Fourier-Mukai Transform).

§ Darboux Type theorems. [Local std. form]

$$\mathbb{R}\times M \xrightarrow{F} M \qquad \left(\text{i.e. } \mathbb{R} \xrightarrow{\text{gp. homo.}} \mathrm{Diff}(M) \atop t \longmapsto F_t = F(t,-)\right)$$

flow



$$v_t := \frac{dF_t}{dt}\circ F_t^{-1}$$

$$F_t^*\omega_t = \omega_0 \iff L_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0$$

$$\left(\text{Pf: } \frac{d}{dt}(F_t^*\omega_t) = F_t^* L_{v_t}\omega_t + F_t^*\frac{d\omega_t}{dt}\right)$$

<u>Moser theorem</u> (M,ω_t) sympl. cpt.

$[\omega_t]$ indep. of $t \Rightarrow \exists F$ s.t. $F_t^*\omega_t = \omega_0$

Note: (McDuff) $[\omega]=[\omega']$ sympl. $\not\Rightarrow$ $F^*\omega' = \omega$

Pf: $-\frac{d\omega_t}{dt} = d\mu_t \quad \exists\, \mu_t \in \Omega^1(M)$ $\left(\because \frac{d}{dt}[\omega_t] = 0\right)$

$= d(\iota_{v_t}\omega_t)$ ∃ v.f. v_t ($\because$ ω_t non-degen.)

$= \mathcal{L}_{v_t}\omega_t$ ($\because d\omega_t = 0$)

i.e. $\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0$

$\overset{\text{integrate}}{\rightsquigarrow}$ $F_t^*\omega_t = \omega_0$ $\exists\, F_t$ QED.

- Similarly, if ν_0, ν_1 are volume forms on M cpt.

$$\int_M \nu_0 = \int_M \nu_1 \iff \varphi\in\mathrm{Diff}(M) \text{ w/ } \varphi^*\nu_1 = \nu_0$$

Theorem. $X \hookrightarrow M$, ω_0, ω_1 both sympl.

$\forall\ p \in X$, $\omega_0(p) = \omega_1(p) \in \Lambda^2 T_p^* M$

$\Longrightarrow$ locally near X, $\exists$ diffeo. φ

s.t. $\varphi^* \omega_1 = \omega_0$ & $\varphi|_X = id$

Take $X = \{p\}$, we have

Cor (Darboux Lemma) $p \in (M, \omega)$

$\Longrightarrow$ locally near p, $(M, \omega) \simeq (\mathbb{R}^{2n}, \omega_{std})$.

Pf: Linear alg. $\Rightarrow (T_pM, \omega_p) \simeq (\mathbb{R}^{2n}, \omega_{std})$

Thm. $\Rightarrow$ locally the same.

Take $X = L \overset{\text{Lagr.}}{\subset} M$, we have

Cor. (Weinstein) $L \overset{\text{Lagr.}}{\subset} (M, \omega)$

$\Longrightarrow$ locally near L, $(M, \omega) \simeq (T^*L, \omega_{can})$.

Equivalent, $L \subset M$ Lagr. w.r.t. ω_0 & ω_1,

$\Rightarrow \exists$ nbd. & loc. diffeo. φ fixing L, s.t.

$\varphi^* \omega_1 = \omega_0$.

Pf. $\omega_1|_L = \omega_0|_L = 0 \in \Gamma(\Lambda^2 T_L^*)$

To apply the thm., we need, $\forall\ p \in L$

$\omega_1(p) = \omega_0(p) = 0 \in \Gamma(\Lambda^2 T^*_{\underset{\sim}{M}, p})$

Pick any splitting $TM|_L = TL \oplus N_{L/M}$ of

$$\begin{tikzcd}
0 \arrow[r] & TL \arrow[r] & TM|_L \arrow[r] & N_{L/M} \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

(say by using any metric.)

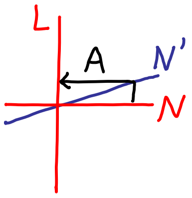
$$\begin{tikzcd}
(TM|_L, \omega_0) \arrow[r, "\Phi", "\sim"'] \arrow[d, equal] & (TM|_L, \omega_1) \arrow[d, equal] \\
TL \oplus N'_{L/M} \arrow[r, "\begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix}"] & TL \oplus N''_{L/M}
\end{tikzcd}$$

$\Longrightarrow$ $\exists$ canon. Φ

(i.e. id. on L)

reason: linear alg.

v.s. $L \oplus N \subset (M, \omega)$ w/ L Lagr.



$\Longrightarrow$ $\exists$ canon. Lagr. $N' \subset M$

$L \oplus N' = M$ $\succ$ $N' = \mathrm{Graph}(A : N \to L)$
$\exists A$.

N' Lagr

$\Longleftrightarrow$ $\omega(u + Au, v + Av) = 0 \quad \forall\, u, v \in N$

$\omega(u, v) + \omega(Au, v) + \omega(u, Av) + \cancel{\omega(Au, Av)} = 0$ (∵ L Lagr)

L Lagr $\Rightarrow$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "\cong"'] & M \arrow[r] \arrow[d, "\cong"', "\lrcorner\omega"] & N \arrow[r] \arrow[l, bend right, "\text{split}"'] \arrow[d, "\cong"'] & 0 \\
0 & N^* \arrow[l] & M^* \arrow[l] & L^* \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$A = \frac{-1}{2} \times ($  $)$

For Lagr. $L \subset M$ wrt ω_0 and ω_1.

$\rightsquigarrow$ $A_0, A_1 : N \to L$

$\rightsquigarrow$ $N'_0, N'_1 \subset M$ $\qquad N'_0 \xrightarrow{\lrcorner\omega_0} L^* \xleftarrow{\lrcorner\omega_1} N'_1$

$$\Phi = \begin{pmatrix} id & 0 \\ 0 & \omega_1^{-1} \circ \omega_0 \end{pmatrix} : \underbrace{L \oplus N'_0}_{M} \longrightarrow \underbrace{L \oplus N'_1}_{M}$$

Recall: Whitney ext^n thm. (Diff. Topo.)

$$\forall\ L \overset{\text{submfd}}{\subset} M \qquad \begin{tikzcd} 0 \arrow[r] & TL \arrow[r] \arrow[d, equal] & TM|_L \arrow[d, "\forall \Phi", "\cong"'] \\ 0 \arrow[r] & TL \arrow[r] & TM|_L \end{tikzcd}$$

$$\Longrightarrow \exists\ \varphi \in \mathrm{Diff}(\mathrm{nbd}(L))$$

$$\text{s.t. } \varphi|_L = id. \ \& \ d\varphi|_L = \Phi$$

Hence the Weinstein's result. Q.E.D.

Proof of thm: $\left[\begin{array}{l} \omega_0 = \omega_1 \text{ on } X \subset M \\ \Longrightarrow \begin{tikzcd} (M,\omega_0) \arrow[r, "\sim", "\text{near } X"'] & (M,\omega_1) \\ X \arrow[u, phantom, "\cup"] \arrow[r, equal] & X \arrow[u, phantom, "\cup"] \end{tikzcd} \end{array} \right.$

$\omega_1 = \omega_0$ on $X \overset{\text{homotopy eq.}}{\hookrightarrow} \mathrm{nbd}(X)$ $(\& \ d\omega_0 = 0 = d\omega_1)$

$\Rightarrow \omega_1 = \omega_0 + d\mu$ on $\mathrm{nbd}(X)$

$\exists \mu$ w/ $\mu = 0$ on X

$\omega_t := \omega_0 + t\, d\mu$ sympl. on $\mathrm{nbd}'(X), \forall\, 0 \leq t \leq 1$

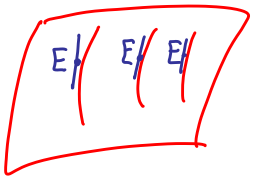
$\rightsquigarrow$ v.f v_t : $\iota_{v_t} \omega_t = -\mu$

integrate $v_t \rightsquigarrow \varphi^* \omega_1 = \omega_0$ in $\mathrm{nbd}''(X)$

(integrate okay since $\mu = 0$ on X). QED.

Remark: sympl. nbd. of isotropic emb. $\leftrightarrow$ sympl. VB.

sympl. nbd. of coisotropic emb $\leftrightarrow$ sympl. nbd. of zero. sect^n



$X \underset{\text{coisot}}{\subset} (M,\omega)$ of $E^* \longrightarrow M$

§ Hamiltonian vector fields.

$\omega \in \Omega^2(M)$ sympl.

$$\overbrace{\Omega^0(M)}^{C^\infty(M)} \xrightarrow{d} \Omega^1(M) = \Gamma(T^*_M) \xleftarrow[\cong]{\iota\omega} \overbrace{\Gamma(T_M)}^{\mathrm{Vect}(M)}$$

$$f \qquad\qquad df \qquad\qquad\qquad\qquad X_f$$

Hamil. v.f.

i.e. $\iota_{X_f}\omega = df$

$$X_f = 0 \iff f \equiv \mathrm{const}$$

Note: $\mathrm{Vect}(M) = \mathrm{Lie\,Diff}(M)$ Lie alg.

$$\rotatebox{90}{$\subseteq$}$$

$\mathrm{Vect}(M,\omega) = \mathrm{Lie\,Diff}(M,\omega)$ sympl. v.f.

$$\rotatebox{90}{$\in$}$$

i.e. X s.t. $\mathcal{L}_X\omega = 0$

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(\cancel{d\omega})$$

$$\xRightarrow{\iota_{X_f}\omega = df} \quad \mathcal{L}_{X_f}\omega = d(df) = 0$$

i.e. $0 \to \mathbb{R} \to C^\infty(M) \to \mathrm{Vect}(M,\omega) \to H^1(M,\mathbb{R}) \to 0$

$$f \longmapsto X_f$$

exact.

$$X \longmapsto [\iota_X\omega]$$

Claim: Exact seq. of Lie alg.

i.e. $[X_f, X_g] = X_{\{f,g\}}$

where $\{f,g\} = X_f(g) \in C^\infty(M)$

Pf. of claim: $\iota_{[X_f,X_g]}\omega \overset{?}{=} d(X_f(g))$

$$\mathrm{LHS} = \iota_{\mathcal{L}_{X_f}(X_g)}\omega \underset{(\because \mathcal{L}_X\omega = 0)}{=\!=} \mathcal{L}_{X_f}(\underbrace{\iota_{X_g}\omega}_{dg}) = d(\underbrace{\mathcal{L}_{X_f} g}_{X_f(g)})$$

$(C^\infty(M), \{\ \ \})$ Lie alg.

- $\{f,g\} = \underbrace{X_f(g)} = -X_g(f) = -\omega(X_f, X_g)$

$(\because\ dg(X_f) = (\iota_{X_g}\omega)(X_f) = \omega(X_g, X_f)\ )$

Leibniz rule : $\{f\,,\, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}$

(Pf: $\{f, gh\} = d(gh)(X_f) = (h\,dg + g\,dh)(X_f)$)

i.e. $\Big(C^\infty(M)\,,\ \underset{\substack{\nearrow \\ \text{commutative} \\ \text{assoc. alg.}}}{\cdot}\ ,\ \underset{\substack{\uparrow \\ \text{Lie alg}}}{\{\ \ \}}\Big)$ Poisson algebra.

Deformation quantization :

$\exists ?\ \star_\hbar$ assoc. product on $C^\infty(M)[[\hbar]]$

s.t. $f \star_\hbar g = f\cdot g + \hbar\{f,g\} + O(\hbar^2)$.

§ Completely Integrable System

Sympl. $(M, \omega) \xrightarrow[H]{} \mathbb{R}$

Noether theorem.

$$\{f, H\} = 0$$

$\Longleftrightarrow$ f is const. along X_H-flow

(i.e. f is a conservation law).

Pf: $\rho_t : \mathbb{R}_t \times M \to M$ flow gen. by X_H.

$$\frac{d}{dt}(f \circ \rho_t) = X_H(\rho_t^*(f)) = \rho_t^*(\underbrace{X_H(f)}_{\{f,H\}=0})$$

Try to find as many conserved quantities as possible:

$(H=) f_1, f_2, \dots, f_k : M^{2n} \to \mathbb{R}$

(1) $\{f_i, f_j\} = 0 \quad \forall\ i, j$

(2) indep. ($\forall$ X_{f_i}'s are l.i. in T_pM, $\forall p \in M$)

$(1)+(2) \Rightarrow \operatorname{Span}(X_{f_i}(p)\text{'s}) \leqslant T_pM$

isotropic of dim k

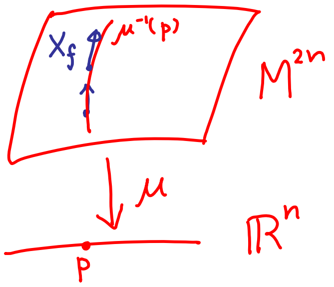
$\Rightarrow k \leqslant n$

Best scenerio : # cons. law $= n = \frac{1}{2} \dim M$

$\mu = (f_1, \dots, f_n) : M^{2n} \longrightarrow \mathbb{R}^n$

Lagrangian bundle / Completely Integ. System.

$X_{f_1}, \dots, X_{f_n}$: vertical vector fields w.r.t. $M^{2n} \xrightarrow{\mu} \mathbb{R}^n$



i.e. $\mu_*(X_{f_j}) = 0 \quad \forall\ j$

$\Big(\because\ X_{f_1}(f_2) = \{f_1, f_2\} = 0$
$\Rightarrow f_2$ is const. along X_{f_1}-flow $\Rightarrow f_{2*}(X_{f_1}) = 0 \Big)$

⇝ On $\mu^{-1}(p)$ ← dim n, $\exists\ n$ vector fields $\{X_{f_j}\}_{j=1}^{n}$

- linearly indep. at every point in $\mu^{-1}(p)$
- commuting

$\xRightarrow[\text{Thm.}]{\text{Frobenius}}$ Can integrate to get affine coord.,
(assume complete v.f.)

$$\mu^{-1}(p) = \frac{\mathbb{R}^n}{\Lambda} = \mathbb{R}^{n-k} \times T^k$$

If μ : proper $\Longrightarrow \mu^{-1}(p) \simeq T^n$ (affine mfd).

Called angle coordinates.

Fact: $\exists$ coord. on $\mathbb{R}^n$ s.t. $\omega = \omega_{std}$,
called action coordinates.

(Eg. Simple pendulum (see p.112)).

Fact: X_H-flow = linear flow on
$\mu^{-1}(p) \simeq \mathbb{R}^n/\Lambda$

§ Hamiltonian mechanics

Newton's 2nd law: $\vec{F} = m\vec{a}$ where $\vec{a} = \frac{d^2}{dt^2}\vec{x}(t) \in \mathbb{R}^3$

Assume $\mathrm{Curl}(\vec{F}) \equiv \vec{\nabla} \times \vec{F} = 0$ i.e. conservative

$\iff$ $\vec{F} = -\vec{\nabla} V$ $\quad \exists V : \mathbb{R}^3 \to \mathbb{R}$ potential

$\iff$ Work $W(a,b) := \int_a^b \vec{F} \cdot d\vec{x}$

well-def$^{\text{d}}$ (i.e. indep. of path $a \overset{\gamma}{\rightsquigarrow} b$)

$\implies$ $H := \frac{1}{2} m |\vec{v}|^2 + V(x)$ is conserved

i.e. $\frac{dH}{dt} = 0$ Conservation Law.

Eg. Gravity : $V(x) = \frac{c}{|x|}$.

Aim : To understand Conservation Law.

Given $q(t) \in \mathbb{R}^3$ Configuration space

$\rightsquigarrow$ $(q(t), \underbrace{p(t)}_{m\frac{dq}{dt}}) \in T^*\mathbb{R}^3$ Phase space

$\omega_{std} = \sum dq^j \wedge dp_j$

$$H = \frac{1}{2m}|p|^2 + V(q) : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}$$

$\rightsquigarrow$ $X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j} \frac{\partial}{\partial p_j} \right)$. $\quad (\iota_{X_H}\omega = dH)$

Ex : $m\frac{d^2q}{dt^2} = F = -\nabla V \qquad \mathbb{R}^3$

$\iff$ $\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}$, $\frac{dp_j}{dt} = -\frac{\partial H}{\partial q^j}$ Hamilton eqt.

$\iff$ $(q(t), p(t)) \in T^*\mathbb{R}^3$ integral curve of X_H

§ Lagrangian mechanics.

e.g. n particles w/ constraints

$$\gamma(t) \in \underset{\substack{\text{constaint}\\ \text{submfd.}}}{X} \subset \underset{\uparrow_{n \text{ particles}}}{\mathbb{R}^{3n}}$$

$$\underbrace{A(\gamma)}_{\text{Action}} := \int_a^b \Bigg[\underbrace{\sum_{j=1}^{n} \frac{m_j}{2} \left| \frac{d\gamma_j(t)}{dt} \right|^2 - V(\gamma(t))}_{\mathcal{L}(\gamma(t),\, \dot{\gamma}(t))} \Bigg] dt$$

$A : \mathcal{L}X \to \mathbb{R}$

$\mathcal{L} : TX \longrightarrow \mathbb{R}$ Lagrangian

- Critical point of A

$$\Longleftrightarrow \quad \frac{\partial \mathcal{L}}{\partial x^j} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v^j} \text{ along } \gamma(t)$$ (Euler-Lagr. Eqt.)

- When $X = \mathbb{R}^{3n}$ (i.e. $\nexists$ constraint)

  EL eqt. $\Longleftrightarrow$ $ma = F = -\nabla V$.

- When $V \equiv 0$

  i.e. $\mathcal{L}(x,v) = \frac{1}{2}|v|^2 : TX \longrightarrow \mathbb{R}$

  $= \frac{1}{2} g_{ij}(x) v^i v^j$

  i.e. $A(\gamma) = \frac{1}{2} \int \left| \frac{d\gamma}{dt} \right|^2 dt$

  EL eqt. $\Longleftrightarrow$ geodesic eqt.

$$\frac{d^2\gamma^\ell}{dt^2} + \Gamma^\ell_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0$$

where $\Gamma^\ell_{ij} = \frac{1}{2} g^{\ell k} (g_{kj,i} + g_{ki,j} - g_{ij,k})$

§ Legendre transform.

Lagr. mechanics Hamil. mechanics

$\mathcal{L} : TX \longrightarrow \mathbb{R}$ $H : T^*X \longrightarrow \mathbb{R}$

⇝ EL eqt. $\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}$ along γ $\frac{dq}{dt} = \frac{\partial H}{\partial p}$, $\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$ Hamilton eqt.

$$\forall\ q \in M, \quad \begin{array}{ccc} T_q X & \longrightarrow & T_q^* X \\ v^j & \longmapsto & p_j = \frac{\partial \mathcal{L}}{\partial v^j} \end{array}$$

($\mathcal{L}|_{T_qX}$ strictly convex $\Rightarrow$ 1−1)

$$\mathcal{L}(q,v) : TX \to \mathbb{R} \leftrightsquigarrow H(q,p) = p \cdot v - \mathcal{L}(q,v)$$

Geometric explanation. $V = T_q X$

1) $T^*V = V \times V^* = T^*V^*$

$$\omega_{T^*V} = \!=\!=\!=\!= (-1) \cdot \omega_{T^*V^*}$$

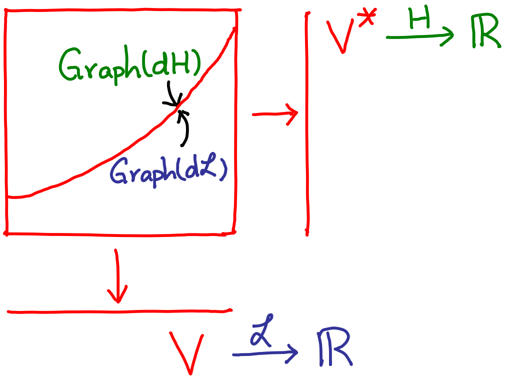
$\Rightarrow$ same Lagrangian submfds.

2) $\mathcal{L} : V \to \mathbb{R}$

⇝ $\mathrm{Graph}(d\mathcal{L}) \underset{\text{Lagr}}{\subset} T^*V$

$\qquad\qquad ||$

$\quad\ \mathrm{Graph}(dH) \underset{\text{Lagr}}{\subset} T^*V^*$



Claim: (1) $V \to \mathrm{Graph} \longrightarrow V^*$ is $v \mapsto p = \frac{\partial \mathcal{L}}{\partial v}$

(2) $H(p) = p\, v - \mathcal{L}(v)$

Pf $\mathrm{Graph}(d\mathcal{L}) = \{ p_j = \frac{\partial \mathcal{L}}{\partial v^j} \} \Rightarrow$ (1)

$$\underbrace{\omega_{T^*V}}_{d\alpha_{T^*V}} = \sum dp_j \wedge dv^j = -\underbrace{\omega_{T^*V^*}}_{d\alpha_{T^*V^*}}$$

$$\alpha_{T^*V} = \sum p_j \, dv^j \qquad \alpha_{T^*V^*} = \sum v^j \, dp_j$$

$$= d\mathcal{L} \text{ on Graph} \qquad = dH \text{ on Graph}$$

$$\alpha_{T^*V} + \alpha_{T^*V^*} = d\left(\sum p_j v^j\right)$$

$$\Rightarrow \mathcal{L} + H = p v \text{ on Graph} \Rightarrow (2).$$

(up to const.)

Claim: Under Legendre transf.,

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} \Longleftrightarrow \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

along $(q, v) = (\gamma(t), \dot{\gamma}(t)) \in TX$ (i.e. $v = \frac{dq}{dt}$) along $(q, p) \in T^*X$

Pf: $H(q, p) = p \cdot v - \mathcal{L}(q, v)$

$$\frac{\partial H}{\partial p} = v + p \frac{\partial v}{\partial p} - \frac{\partial \mathcal{L}}{\partial v}\frac{\partial v}{\partial p}$$

$$= \frac{dq}{dt} + \left(p - \frac{\partial \mathcal{L}}{\partial v}\right)\frac{\partial v}{\partial p}$$

So, $p = \frac{\partial \mathcal{L}}{\partial v} \Longleftrightarrow \frac{\partial H}{\partial p} = \frac{dq}{dt}$

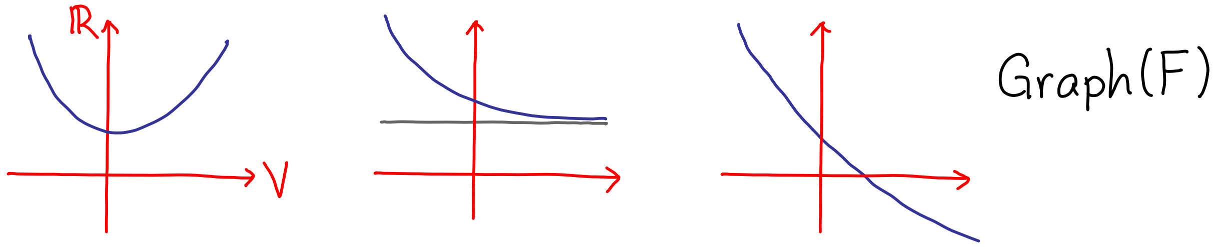
$$\frac{\partial H}{\partial q} + \underbrace{\frac{\partial H}{\partial p}}_{\frac{dq}{dt} = v}\frac{\partial p}{\partial q} = \frac{\partial p}{\partial q} \cdot v - \frac{\partial \mathcal{L}}{\partial q}$$

$$\Rightarrow \frac{\partial H}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}$$

So $\frac{\partial H}{\partial q} = -\frac{\partial p}{\partial t} \Longleftrightarrow \frac{\partial \mathcal{L}}{\partial q} = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v}\right).$

§ Legendre transform, revisited.

$F : \underbrace{V}_{\cong \mathbb{R}^n} \longrightarrow \mathbb{R}$ strictly convex



$\exists$ critical point of F

$\iff \exists$ local minimal

$\iff \exists!$ critical pt. (a global min.)

$\iff F$ proper : ($p \to \infty$ in V $\Rightarrow$ $F(p) \to \infty$ in $\mathbb{R}$)

Call F Stable.

Given F $\rightsquigarrow$ Legendre transf.

$$L_F : V \longrightarrow V^*$$
$$x \longmapsto dF(x)$$

$$F \text{ st. convex} \Rightarrow L_F : V \underset{\text{diffeo.}}{\xrightarrow{\sim}} \underbrace{\mathrm{Image}(L_F)}_{V_s^*} \subseteq V^*$$

Thm. (1) $\ell \in V_s^*$

$\iff F_\ell := F - \ell : V \to \mathbb{R}$ stable

(2) $L_F(x_0) = \ell \iff x_0$ is crit. pt. of F_ℓ.

Define $F^* : V_s^* \longrightarrow \mathbb{R}$

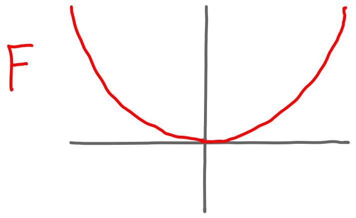
$$F^*(\ell) = -F_\ell(x_0) = -\min_V F_\ell$$

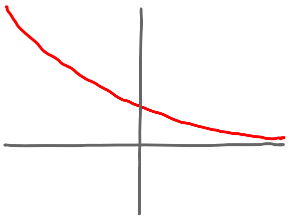
i.e. $F^*(\ell) = \ell(x_0) - F(x_0)$ w/ $dF(x_0) = \ell$

Ex. $V_s^* \subset V^*$ convex.

Eg. $n=1$. F convex $\Leftrightarrow$ $F' \nearrow$

$\Rightarrow$ $p \mapsto F'(p)$ diffeo. on image

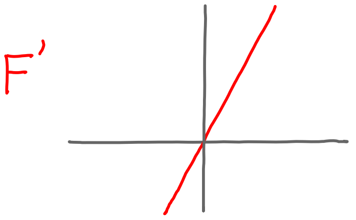


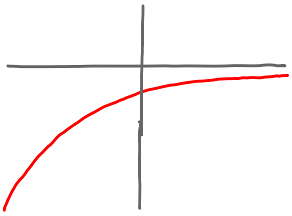


$F(x) = x^2$
$F' = 2x$

$F(x) = e^{-x}$
$F' = -e^{-x}$

$(+\ell x)$
$(+\ell)$



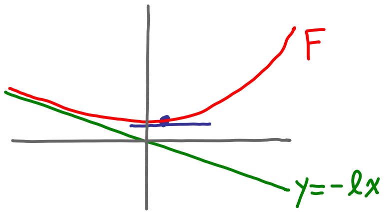


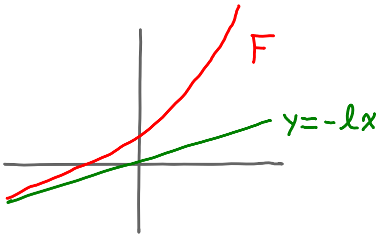
$F' : \mathbb{R} \xrightarrow{\cong} \mathbb{R}$

$F' : \mathbb{R} \xrightarrow{\cong} (-\infty, 0)$ $(-\infty, \ell)$

Eg. $F(x) = e^x - \ell x$, $F' = e^x - \ell$

Eg.





$\ell > 0$
stable

$\ell < 0$
not stable

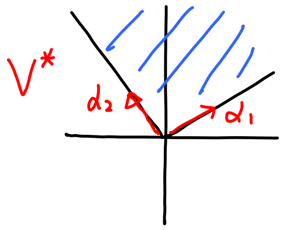
- $F(x)$ $(\geq)$ quadratic growth at ∞

  $\Rightarrow \quad \operatorname{Im} L_F = V^*$

- $F(x) = \sum_i c_i e^{\alpha_i(x)}$ w/ $\alpha_i \in V^*$

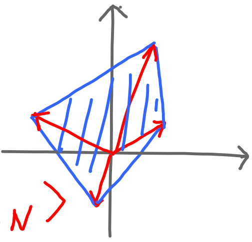
  $\forall\ c_i > 0$ $(\Rightarrow F$ convex$)$

  $\Rightarrow \quad \operatorname{Im} L_F = \operatorname{Cone}\langle \alpha_1, \cdots, \alpha_N \rangle$



- $F(x) = \log\left(\sum_i c_i e^{\alpha_i(x)}\right)$

  $\Rightarrow \quad \operatorname{Im} L_F = \operatorname{Convex\ Hull}\langle \alpha_1, \cdots, \alpha_N \rangle$



- Recall $\quad L_{F^*} \circ L_F = 1_V : V \longrightarrow V$

  $F^{**} = F : V \longrightarrow \mathbb{R}$

In particular, $\forall\ p \in V$, $\forall\ \ell \in V^*_s$

$$F^*(\ell) = -F_\ell(x_0) \geq -\underbrace{F_\ell(p)}_{(F(p)-\ell(p))}$$

i.e. $\underline{F(p) + F^*(\ell) \geq \ell(p)}$

Eg. $F(x) = \frac{1}{p}|x|^p$ w/ $p > 1$ $(\Rightarrow$ convex$)$

$\Rightarrow F^*(y) = \frac{1}{q}|y|^q$ w/ $\frac{1}{p} + \frac{1}{q} = 1$

$\left(\because\ y = F'(x) = x^{p-1},\ F^*(y) = yx - F(x) = y \cdot y^{\frac{1}{p-1}} - \frac{1}{p} y^{\frac{p}{p-1}} = \frac{y^q}{q}\right)$

$\xRightarrow{\text{Claim}} \forall\ a, b > 0$, $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$ (Young ineqt.)

§ Moment Maps.

$$G \overset{\rho}{\curvearrowright} (M, \omega)$$

$$\Longleftrightarrow \quad G \xrightarrow{\rho} \mathrm{Diff}(M, \omega) \equiv \mathrm{Symp}(M)$$

Def: Hamiltonian action if $\exists\, \mu^\#$

$$\begin{tikzcd}
 & & C^\infty(M) \arrow[d] & f \arrow[d, mapsto] & \{f, g\} = X_f(g) \arrow[d] \\
\Longrightarrow & \mathfrak{g} \arrow[r, "d\rho"'] \arrow[ur, dashed, "\underset{\text{(Lie homo).}}{\nexists}\ \mu^\#"] & \mathrm{Vect}(M, \omega) \arrow[d] & X_f & {[\ ,\ ]} \\
 & & H^1(M, \mathbb{R}) \arrow[d] & \iota_{X_f}\omega = df & \\
 & & 0 & &
\end{tikzcd}$$

eg. $G = S^1$, $\quad \mathfrak{g} = \mathbb{R}\langle X = \frac{\partial}{\partial \theta} \rangle$

$$0 = \mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X(\cancel{d\omega})$$

Hamil. $\Longleftrightarrow$ $\iota_X \omega = d\mu \quad \exists\, \mu : M \longrightarrow \mathbb{R}$

(being Lie homo. is automatic).

$$\Longleftrightarrow \quad (d + \iota_X)(\omega - \mu) = 0$$

$$\Longleftrightarrow \quad (d + \iota_X)\, e^{\omega - \mu} = 0$$

$\mu^\# : \mathfrak{g} \longrightarrow C^\infty(M)$ Lie alg. homo.

$\overset{\text{(Ex)}}{\Longleftrightarrow}$ $\mu : M \longrightarrow \mathfrak{g}^*$ G-equivar.

$$\mu^\#(X)(x) = \mu(x)(X) \in \mathbb{R}$$

(where $G \curvearrowright \mathfrak{g}^*$ via coadj. action).

Eg. (Angular momentum).

$$O(3) = \mathrm{Aut}(\mathbb{R}^3, \langle\ \rangle_{\mathrm{std}}) \curvearrowright \mathbb{R}^3$$ rotation.
& reflection.

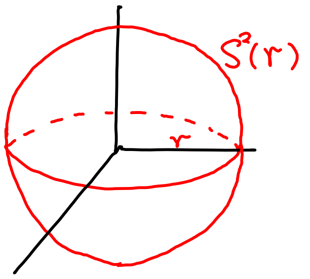
This is the (co-) adjoint repr. $(\mathbb{R}^3 \xrightarrow[\cong]{\lrcorner g} \mathbb{R}^{3*})$

$$\left(\begin{array}{l} \text{i.e. } \underline{\mathfrak{o}}(3),\ [\ ,\ ] \longleftrightarrow \mathbb{R}^3,\ \times \\ \quad \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \underline{\mathfrak{o}}(3) \qquad (a,b,c) \in \mathbb{R}^3 \end{array} \right.$$

$$\left(\begin{array}{l} \left.\frac{d}{dt}\right|_{t=0} \left(\langle e^{tA}u, e^{tA}v \rangle = \langle u, v \rangle \right) \\ \qquad \langle Au, v \rangle + \langle u, Av \rangle = 0 \\ A + A^T = 0, \text{ i.e. } A \text{ skew symmetric} \end{array} \right.$$

(co-) adjoint orbits :

$$\{0\},\ S^2(r) \subset \mathbb{R}^3$$



$$\underbrace{\underline{\mathfrak{o}}(3)}_{\mathbb{R}^3} \curvearrowright \underbrace{T^*\mathbb{R}^3}_{\mathbb{R}^3 \oplus \mathbb{R}^3} \qquad \omega_{\mathrm{std}}$$

$$a \cdot (X, Y) = (a \times X, a \times Y)$$

(Ex) moment map is $\mu(X, Y) = X \times Y$.

Namely, the moment map for rotation is angular momentum.

Similarly, the moment map for translation is linear momentum, $\mu_{\mathrm{translation}}(X, Y) = Y$.

§ Coadjoint orbit

$$G \curvearrowright G \quad \text{Conjugate}$$

$$g \qquad a \mapsto g \cdot a \cdot g^{-1}$$

$$e \mapsto g e g^{-1} = e$$

linearize at $e \rightsquigarrow$

$$G \curvearrowright T_eG = \mathfrak{g} \quad \text{adjoint action}$$

i.e. $Ad : G \longrightarrow GL(\mathfrak{g})$ $\Big(\Rightarrow \underbrace{d(Ad)}_{[\ ,\ ]} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$

$\rightsquigarrow$ $Ad^* : G \longrightarrow GL(\mathfrak{g}^*)$ coadj. action

$$\langle Ad^*(g)\xi , X \rangle = \langle \xi , Ad(g^{-1}) X \rangle$$

Coadj. orbit : $\mathcal{O}_\xi := G \cdot \xi \subset \mathfrak{g}^*$

Claim: 1° $(\mathcal{O}_\xi , \omega_\xi)$ Symplectic

2° Hamiltonian $G \curvearrowright \mathcal{O}_\xi \xhookrightarrow{\mu} \mathfrak{g}^*$

moment map = natural inclusion.

3° Hamiltonian $G \curvearrowright M$, transitive $\Rightarrow M = \mathcal{O}_\xi$

Eg. $U(n) = Aut(\mathbb{C}^n , \langle \ \rangle_{std})$

$$u(n) = \{ A : A + \bar{A}^T = 0 \}$$

$$= i \{ B : B - \bar{B}^T = 0 \} =: i\, Herm_n$$

i.e. Hermitian symmetric

(Just neglect i)

$u(n) \simeq u(n)^*$ via inner product $Tr\, X Y$. ($\because$ cpt)

Coadj. action $U(n) \curvearrowright u(n)^* \simeq \mathrm{Herm}_n$

$$A \cdot \xi = A \xi A^{-1}$$

$$\mathcal{O}_\xi = \mathcal{O}_\eta \iff \mathrm{Spec}(\xi) = \mathrm{Spec}(\eta)$$

i.e. same eigenvalues.

Cor: $u(n)^* = \mathrm{Herm}_n \supseteq \{\text{diagonal}\} = t^*$

$\Rightarrow \mathcal{O}_\xi \cap t^*$ unique up to permuting eigenvalues by $S_n = W$ (Weyl gp.).

Sympl. form ω on $\mathcal{O}_\xi \subset u(n)^*$:

$$\omega(\eta)(X, Y) = i \,\mathrm{Tr}([X, Y]\eta)$$

Write $\lambda = \mathrm{Spec}(\xi) \in \mathbb{R}^n / S_n$
$= (\lambda_1, \lambda_2, \cdots\cdots, \lambda_n)$

$\lambda = (1\ 1 \cdots 1) = (1^n) \Rightarrow \mathcal{O}_\lambda = \{pt\}$

$\lambda = (1, 2^{n-1})$. $\xi \in \mathcal{O}_\lambda \Rightarrow$

$$\mathbb{C}^n \overset{\text{eigenvalue}}{\underset{\text{decomp.}}{=\!=}} \underbrace{\mathrm{Ker}(\xi - I)}_{\dim 1} \overset{\perp}{\oplus} \underbrace{\mathrm{Ker}(\xi - 2I)}_{\dim (n-1)}$$

i.e. $\mathcal{O}_\lambda \cong \mathbb{CP}^{n-1} = \dfrac{U(n)}{U(1)\,U(n-1)}$

$\lambda = (1^{n_1} 2^{n_2} 3^{n_3} \cdots)$ $\qquad \mathcal{O}_\lambda = \dfrac{U(n)}{U(n_1)U(n_2)U(n_3)\cdots}$

$\Sigma n_i = n$

partial flag variety.

$\mathrm{Gr}_{\mathbb{C}}(r,n)$ Complex Grassmannian
$\begin{array}{c}\mathrm{Gr}_{\mathbb{C}}(r,n)\\ \cong \\ \mathcal{O}_{\lambda=(1^r 2^{n-r})}\end{array} \overset{\mu}{\hookrightarrow} \mathfrak{u}(n)^* = \mathrm{Herm}_n$

Explicitly,

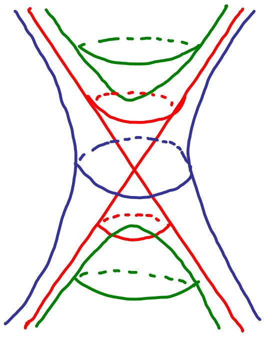
$\mathbb{C}^r \simeq P \leq \mathbb{C}^n$

$\rightsquigarrow$ choose o.n. basis of P : $v_1, \dots, v_r$

$\Rightarrow \mathcal{J}_P = \left(v_1, \dots, v_r\right) \cdot \left(v_1, \dots, v_n\right)^* \in \mathrm{Herm}_n$

(indep. of choice of v_i's s.t. $AA^* = I$ for $A \in U(r)$)

Remark: Coadjoint orbits for $SL(2,\mathbb{R})$



Aim: $G \curvearrowright (\mathcal{O}_\xi, \omega_\xi) \overset{\mu}{\hookrightarrow} \mathfrak{g}^*$

- $\mathfrak{g}^*$ has naturnal Poisson structure

Recall: M Poisson mfd

$\Leftrightarrow (C^\infty(M), \cdot, \{\ \})$ Poisson alg.

Leibniz $(\{f, gh\} = \{f,g\}h + g\{f,h\})$

$\Rightarrow \{f,g\} = \pi(df \wedge dg) \ \exists \ \pi \in \Gamma(\Lambda^2 T_M)$

[eg. Sympl. $\Rightarrow$ Poisson

$\omega = \pi$ via $T^* \xleftarrow[\cong]{\lrcorner\,\omega} T$

$\exists$ natural $\pi \in \Gamma(\underset{\substack{\cup \\ \xi}}{\mathfrak{g}^*}, \Lambda^2 T\mathfrak{g}^*)$

$$\pi(\xi) \in \Lambda^2 T_{\mathfrak{g}^*,\xi} = \Lambda^2\mathfrak{g}^*$$

i.e. $\pi(\xi) : \Lambda^2\mathfrak{g} \longrightarrow \mathbb{R}$

$$\pi(\xi)(X \wedge Y) := \xi([X, Y])$$

(namely, $[\ \ ] : \mathfrak{g}\wedge\mathfrak{g} \longrightarrow \mathfrak{g}$

$\rightsquigarrow \quad \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^* \rightsquigarrow \pi$)

Ex: $(\mathfrak{g}^*, \pi)$ Poisson mfd.

(i.e. Jacobi id. for $\{f, g\} = \pi(df \wedge dg)$ ✓)

Indeed $C^\infty(\mathfrak{g}^*) \supset \{\text{poly.}\} = S^\bullet(\mathfrak{g}^*)^* = S^\bullet\mathfrak{g}$

$\supset \{\text{linear}\} = (\mathfrak{g}^*)^* = \mathfrak{g}$

$\{\ \ \}$ on $C^\infty(\mathfrak{g}^*)$ restricts to $\{\text{linear}\}$ is just

$[\ \ ] : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$, the Lie bracket.

It has a natural ext^n to $\{\text{poly.}\}$

$$\{\ \ \} : \Lambda^2(S^\bullet\mathfrak{g}) \longrightarrow \mathfrak{g}.$$

At $\xi \in \mathfrak{g}^*$

$$\lrcorner\, \Pi(\xi) : \underbrace{T_\xi^* \mathfrak{g}^*}_{\mathfrak{g}} \longrightarrow \underbrace{T_\xi \mathfrak{g}^*}_{\mathfrak{g}^*}$$

<u>Claim</u>: $\mathrm{Ker}(\lrcorner\, \Pi(\xi)) = \{X \in \mathfrak{g} \mid ad_X^*(\xi) = 0\} \subseteq \mathfrak{g}$

Remark: $\forall$ (M, Π) Poisson mfd.

(1) if $\forall\ \xi \in M$, $\lrcorner\,\Pi(\xi) : T_\xi^* M \xrightarrow[\cong]{} T_\xi M$ isom

then (M, ω) Symplectic (as before)

(2) Say $\dim(\mathrm{Ker}(\lrcorner\,\Pi(\xi)))$ const. (indep. of ξ)

then $\mathrm{Im}(\lrcorner\,\Pi(\xi)) \leq T_\xi M$ <u>integ</u>. distribution,

and each leaf is symplectic.

(3) In general,

$M = \coprod$ (sympl. leaves of different dim.)

Eg. $\Pi = 0 \Rightarrow$ each pt. is a sympl. leaf.

Eg. $(\underline{o}(3) \simeq \mathbb{R}^3, \Pi) \Rightarrow$ sympl. leaves are

2-spheres $S^2(r)$ or origin $\{0\}$.

Pf. of Claim: $X \in \mathrm{Ker}(\lrcorner\,\Pi(\xi))$

$\Leftrightarrow \forall Y \in \mathfrak{g}$, $0 = \Pi(\xi)(X, Y) \triangleq \xi([X, Y])$

$= \xi(ad_X(Y)) = -ad_X^*(\xi)(Y)$

$\Leftrightarrow$ $ad_X^*(\xi) = 0$

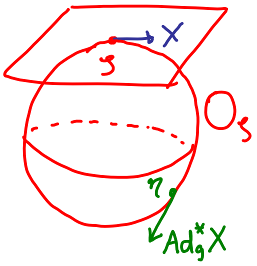
i.e. Π non-degen. along coadj. orbit $\mathcal{O}_\xi$.

Claim ⇒ On $\mathcal{O}_\xi \subset \mathfrak{g}^*$

$$T^*_\xi \mathcal{O}_\xi \overset{\lrcorner \Pi(\xi)}{\underset{\lrcorner \omega(\xi)}{\overset{\sim}{\rightleftarrows}}} T_\xi \mathcal{O}_\xi$$

$\rightsquigarrow$ $\omega \in \Omega^2(\mathcal{O}_\xi)$ non-degen.

Indeed, $\forall\ X, Y \in T_\xi \mathcal{O}_\xi \subseteq \mathfrak{g}^*$



At any $\eta = Ad^*_g(\xi) \in \mathcal{O}_\xi$

$Ad^*_g(X), Ad^*_g(Y) \in T_\eta \mathcal{O}_\xi$

$\omega(\eta)($ —"— , —"— $)$

$= \omega(\xi)(X, Y) = \xi([X, Y])$

($\because$ Π is G-inv.)

Want $\mathcal{O}_\xi \subset \mathfrak{g}^*$

$\equiv G \curvearrowright \mathcal{O}_\xi \xrightarrow{\mu} \mathfrak{g}^*$

i.e. $\forall\ X \in \mathfrak{g}$, $\iota_X \omega \overset{?}{=} -d\varphi_X$

where $\varphi_X : \mathcal{O}_\xi \subset \mathfrak{g}^* \xrightarrow{\langle -, X \rangle} \mathbb{R}$

Pf: $\varphi_X(Ad^*_g(\xi)) = \langle Ad^*_g(\xi), X \rangle$,

$d\varphi_X(\xi)(Y) = \frac{d}{dt}\Big|_{t=0} \varphi_X(Ad^*_{e^{tY}}(\xi))$, $\forall Y \in \mathfrak{g}$

$= \langle ad^*_Y(\xi), X \rangle = -\langle \xi, \underbrace{ad_Y(X)}_{[Y,X]} \rangle$

$= -\iota_X \omega(Y)$

Π : G-inv.

$\Rightarrow \mathcal{L}_X \omega = 0$ on $\mathcal{O}_\xi$ $\quad \forall\, X \in \mathfrak{g}$

$\iota_X d\omega + \cancel{d\,\iota_X\omega} = 0$

$(\because \iota_X\omega = -d\varphi_X)$

$\Rightarrow \iota_X d\omega = 0 \quad \forall\, X \in \mathfrak{g}$

$\Rightarrow d\omega = 0 \quad (\because G \curvearrowright \mathcal{O}_\xi$ transitive$)$.

i.e. $(\mathcal{O}_\xi, \omega)$ Symplectic !

and $G \curvearrowright \mathcal{O}_\xi \overset{\mu}{\hookrightarrow} \mathfrak{g}^*$

(Fact: G compact $\Longrightarrow \pi_1(\mathcal{O}_\xi) = 0$.)

Conversely, if transitive $G \curvearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$,

$\xRightarrow{\text{(G-equivar)}}$ transitive $G \curvearrowright \mu(M) \subset \mathfrak{g}^*$

$\Longrightarrow \mu(M) = \mathcal{O}_\xi$ and $\omega_M = \mu^* \omega_{\mathcal{O}_\xi}$

$\xRightarrow{\omega \text{ non-deg.}}$ $M \xrightarrow{\mu} \mathcal{O}_\xi$ unramif. cover.

$\xRightarrow{\pi_1 \mathcal{O}_\xi = 0}$ $M \overset{\mu}{=\!=} \mathcal{O}_\xi$. !

Theorem $G \curvearrowright (M,\omega)$

(1) $H^1(\mathfrak{g})=0 \Rightarrow \mu$ unique (if exists)

(2) $H^1(\mathfrak{g}) = H^2(\mathfrak{g})=0 \Rightarrow \exists!\ \mu$.

Remark (i) $H^1(\mathfrak{g})=0 \Longleftrightarrow [\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$

(ii) G compact $\Rightarrow$ $H^*(\mathfrak{g}) = H^*_{dR}(G)$.

(iii) G compact simple $\Rightarrow H^1(\mathfrak{g}) = H^2(\mathfrak{g})=0$

(e.g. $G=SU(n), SO(n)$)

Pf. of uniqueness (1).

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "\Phi"] & \mathrm{Vect}(M,\omega) \arrow[r] & H^1(M) \arrow[r] & 0 \\
& & & \mathfrak{g} \arrow[u, "d\rho"'] \arrow[ul, dashed, "\mu^\#"] \arrow[ul, phantom, "\circlearrowleft"', shift right=3]
\end{tikzcd}$$

μ_1, μ_2 2 moment maps

$\Longrightarrow \Phi(\mu_1^\#) = d\rho = \Phi(\mu_2^\#)$

$\overset{\text{exactness}}{\Longrightarrow} \mu_1^\# - \mu_2^\# : \mathfrak{g} \longrightarrow \mathbb{R}$

$\overset{[\ ]_{\mathbb{R}}=0}{\Longrightarrow} (\mu_1^\# - \mu_2^\#)([\mathfrak{g},\mathfrak{g}]) = 0$

$[\mathfrak{g},\mathfrak{g}]=\mathfrak{g} \Longrightarrow \mu_1^\# = \mu_2^\#$ #

Lie algebra cohomology.

$$[\ \ ]:\ \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\rightsquigarrow\ \delta:\ \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^*$$

$$\rightsquigarrow\ \delta:\ \Lambda^k\mathfrak{g}^* \longrightarrow \Lambda^{k+1}\mathfrak{g}^*$$

$$(\delta c)(X_0, X_1, \dots, X_k) = \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

$\delta^2 = 0$ ($\Longleftrightarrow$ Jacobi identity).

$$H^k(\mathfrak{g}) := \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta}\bigg|_{\Lambda^k\mathfrak{g}^*}.$$

In particular, $H^1(\mathfrak{g}) = 0 \overset{\text{Ex.}}{\Longleftrightarrow} [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$

Pf. of (2) in thm.

$$\begin{tikzcd}
 & & & \mathfrak{g} \arrow[d, "dp"] & & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r, "\Phi"] & \mathrm{Vect}(M,\omega) \arrow[r] & H^1(M) \arrow[r] & 0 \\
 & & & Y, Z & &
\end{tikzcd}$$

$d(\omega(Y,Z)) = d(\iota_Y \iota_Z \omega)$ up to signs

$= \mathcal{L}_Y(\iota_Z\omega) + \iota_Y(\cancel{d\iota_Z\omega})$ $\because \mathcal{L}_Z\omega = 0$, $d\omega = 0$

$= \iota_{\mathcal{L}_Y Z}\,\omega + \iota_Z(\cancel{\mathcal{L}_Y\omega})$ $\because \mathcal{L}_Y\omega = 0$

$= \iota_{[Y,Z]}\,\omega$

$\Rightarrow [Y,Z] = X_{\omega(Y,Z)} \in \operatorname{Im}(C^\infty(M) \to \mathrm{Vect}(M,\omega))$

$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \Rightarrow \operatorname{Im}(dp) \subseteq \operatorname{Im}(C^\infty(M) \to \mathrm{Vect}(M,\omega))$

$\Rightarrow \exists$ linear lift $\tau: \mathfrak{g} \longrightarrow C^\infty(M)$ of dp

Want $\tau + b$ Lie alg homo. $\exists\, b: \mathfrak{g} \to \mathbb{R}$ i.e. $b \in \mathfrak{g}^*$.

In general,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathfrak{g}_1 \arrow[r] & \mathfrak{g}_2 \arrow[r] & 0 \\
& & & \mathfrak{g} \arrow[ul, "\tau\ \text{linear}"'] \arrow[u, "\rho\ \text{Lie}"'] \arrow[ul, phantom, shift right=4, "\circlearrowright"]
\end{tikzcd}$$

central extension

Claim: $H^2(\mathfrak{g}) = 0 \Rightarrow \exists\, b \in \mathfrak{g}^*$, $\tau + b$ Lie homo.

Pf. of claim: Define $c : \Lambda^2 \mathfrak{g} \longrightarrow \mathbb{R}$

$$c(X,Y) \triangleq \tau([X,Y]) - [\tau(X), \tau(Y)]$$

- $c(X,Y) \in \mathbb{R}$ $\because$ τ lift ρ & ρ Lie homo.
- $c = 0 \iff \tau$ Lie homo.
- Jacobi id. $\Rightarrow$ $\delta c = 0$, i.e. $[c] \in H^2(\mathfrak{g}) \overset{\text{assumption}}{=} 0$

$\Rightarrow$ $c = \delta b$ $\quad \exists\ b : \mathfrak{g} \longrightarrow \mathbb{R}$

$\Rightarrow$ $\tau + b : \mathfrak{g} \longrightarrow \mathfrak{g}_1$ Lie homo.

compact connected $G \curvearrowright M$

$$\begin{tikzcd}
\Omega^*(M)^G \arrow[r, hook, shift left, "\iota"] & \Omega^*(M) \arrow[l, shift left, "\text{average}"]
\end{tikzcd}$$

$$\int_G (\underset{\uparrow\,\text{left translation.}}{L_g}\varphi)\, dg \longleftarrow\!\shortmid\ \varphi$$

$$\Longrightarrow\ H^*(\Omega^\bullet(M)^G, d) = \overbrace{H^*(\Omega^\bullet(M), d)}^{H^\bullet_{dR}(M)} \quad \text{(Ex.)}$$

G & M cpt. conn. $\xRightarrow{\text{Hodge th.}}$ Harmonic forms are G-inv.

$$H^\bullet_{dR}(G) = H^\bullet(\underbrace{\Omega^\bullet(G)^{G_L}}_{\Lambda^\bullet \mathfrak{g}^*}, d) \qquad (G \text{ cpt})$$

$$= H^\bullet(\mathfrak{g})$$

$$H^\bullet_{dR}(G) = H^\bullet(\Omega^\bullet(G)^{G_L \times G_R}, d) = (\Lambda^\bullet \mathfrak{g}^*)^{Ad\,G}$$

Reason: $d = 0$ on $(\Lambda^\bullet \mathfrak{g})^{Ad_G}$

$$\imath(g) = g^{-1} \Rightarrow \imath^* : (\Lambda^\bullet \mathfrak{g}^*)^{Ad_G} \circlearrowleft$$

$$\imath^*|_{\mathfrak{g}} = -1 \Rightarrow \imath^*|_{\Lambda^k \mathfrak{g}^*} = (-1)^k$$

$$\imath^* d = d\,\imath^* \longrightarrow d = -d \Rightarrow d = 0$$

Remark: G compact

$$\begin{tikzcd} G \arrow[r] & EG \arrow[r] \arrow[d, phantom, "\wr"] & BG \\ & * & \end{tikzcd}$$

$$\xRightarrow{\text{spectral seq.}} H^\bullet(BG) = (\mathrm{Sym}^\bullet \mathfrak{g}^*)^G$$

§ Symplectic quotients.

$$G \curvearrowright (M, \omega)$$

- M/G Not sympl., need $M/\!/G$ $(\Rightarrow \text{even dim})$

Eg. $G \curvearrowright X \Rightarrow G \curvearrowright (T^*X, \omega_{can} = d\alpha_{can})$

At least, $T^*X/\!/G \overset{\text{should}}{=} T^*(X/G)$

In fact G preserves $\alpha_{can} = p\,dq$

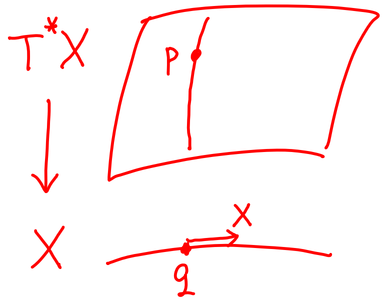
$$\Rightarrow \forall X \in \mathfrak{g} \quad 0 = \mathcal{L}_X \alpha = \iota_X \underbrace{d\alpha}_{\omega \;\Rightarrow} + d\,\underbrace{\iota_X \alpha}_{\mu}$$

i.e. $\mu : T^*X \longrightarrow \mathfrak{g}^*$ moment map

$$\mu(q,p)(X) = \iota_X \alpha\,(q,p)$$

In our case,

$$\iota_X \alpha = \iota_X (p\,dq)$$
$$= \underset{\overset{\Cap}{T_q^*X}}{p}\ \underset{\overset{\Cap}{T_qX}}{X(q)} \in \mathbb{R}$$



$$\Rightarrow \mu^{-1}(0) = \{(q,p) \mid p(X(q)) = 0 \quad \forall X \in \mathfrak{g}\}$$

i.e.
$$\begin{tikzcd} 0 \arrow[r] & \mu^{-1}(0) \arrow[r] & T^*X \arrow[r] \arrow[d] & (T_G X)^* \arrow[r] & 0 \\ & & X & & \end{tikzcd}$$

exact seq. of vector bundles.

$$\Rightarrow \quad \mu^{-1}(0)/G = T^*(X/G) \quad \text{“}= T^*X/\!/G\text{”}.$$

$$G \curvearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}$$

- μ is G-equivar $\Longrightarrow G \curvearrowright \mu^{-1}(0) \subset M$

<u>Claim</u>: $\omega|_{\mu^{-1}(0)}$ can be descended to $\mu^{-1}(0)/G$.

> Recall: $F \to P \xrightarrow{\pi} B$ fiber bundle.
>
> How to characterize $\Omega^\bullet(B) \xhookrightarrow{\pi^*} \Omega^\bullet(P)$?
>
> $\varphi \in \Omega^\bullet(P)$
>
> φ can be descended to B
>
> $\Leftrightarrow \forall\ X \in T_{\text{vert}}P$, $\mathcal{L}_X \varphi = 0 = \iota_X \varphi$
>
> reason: $\varphi = \varphi(\underset{(\because \mathcal{L}_X\varphi = 0)}{\cancel{f_1}}, \dots, b_1, \dots)\, db_1 \cdots \underset{(\because \iota_X\varphi = 0)}{\cancel{df_1}} \cdots$

$\forall\ X \in \mathfrak{g}$, $\mathcal{L}_X \omega = 0$ $\quad$ ($\because$ G preserve ω)

$\iota_X \omega = d\mu^X = 0$ on $\mu^{-1}(0)$

$\Rightarrow \omega|_{\mu^{-1}(0)} = \pi^* \omega_{red}$ on $\mu^{-1}(0) \subset M$

$$\begin{tikzcd} \mu^{-1}(0) \arrow[d, "\pi"] \\ \mu^{-1}(0)/G \end{tikzcd}$$

$\exists$ ω_{red} on $\mu^{-1}(0)/G$

$d\omega_{red} = 0$ $\quad$ ($\because$ $d\omega = 0$)

ω_{red} non-degenerate (prove later).

$M /\!/ G = \mu^{-1}(0)/G$ is called Symplectic Quotient.

(cpt $G \curvearrowright \mu^{-1}(0)$ free $\Rightarrow M /\!/ G$ manifold).

Eg. $\mathbb{C}^{n+1} /\!/ S^1 = \mathbb{C}P^n$.

$$\mathbb{C}^{n+1},\ \omega = dx_0 \wedge dy_0 + \cdots + dx_n \wedge dy_n$$
$$= \frac{i}{2}(dz_0 \wedge d\bar{z}_0 + \cdots + dz_n \wedge d\bar{z}_n)$$

$S^1 \curvearrowright (\mathbb{C}^{n+1}, \omega)$ w/ $e^{i\theta}(z_0, \dots, z_n) = (e^{i\theta} z_0, \dots, e^{i\theta} z_n)$

S^1 preserves ω $\because$ $e^{i\theta} \cdot \overline{e^{i\theta}} = 1$.

$\mathrm{Lie}\, S^1 = \mathbb{R}\langle \underbrace{\frac{\partial}{\partial\theta}}_{X} \rangle$ $\qquad \iota_X \omega = \frac{-1}{2} d\Big(\underbrace{\sum_{j=0}^{n} |z_j|^2}_{|Z|^2} - \underbrace{C}\Big)$ say $C = 2$

$$\Big(\because\ \iota_{\frac{\partial}{\partial\theta}}(r\,dr\,d\theta) = \frac{-1}{2} dr^2\Big)$$

$$\Rightarrow \quad \mu = \frac{1}{2}(|Z|^2 - 1)$$

$$\Rightarrow \quad \mu^{-1}(0) = S^{2n+1}$$

$$\Rightarrow \quad \mathbb{C}^{n+1} /\!/ S^1 = \mu^{-1}(0)/S^1 \cong \mathbb{C}P^n$$

Claim: $\omega_{\mathbb{C}^{n+1}} = \partial\bar{\partial}|Z|^2$

$$\Rightarrow \quad \omega_{\mathbb{C}P^n} = \partial\bar{\partial} \log |Z|^2$$

Say $\omega_{\mathbb{C}P^2} = \partial\bar{\partial}\log(|z_0|^2 + |z_1|^2 + |z_2|^2)$

$$= \underbrace{\partial\bar{\partial}\log|z_0|^2}_{\partial(\cancel{\bar{\partial}\log z_0}) - \bar{\partial}(\cancel{\partial \log \bar{z}_0})} + \partial\bar{\partial}\log\Big(1 + \Big|\frac{z_1}{z_0}\Big|^2 + \Big|\frac{z_2}{z_0}\Big|^2\Big)$$

$$\omega|_{\mathbb{C}^2} = \partial\bar{\partial}\log(1 + |w_1|^2 + |w_2|^2)$$

inhomog. coord. $w_1 = z_1/z_0$, $w_2 = z_2/z_0$

Remark: $H \leq G \rightsquigarrow \mathfrak{h} \hookrightarrow \mathfrak{g} \rightsquigarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$

$$\begin{tikzcd}
 & & & \mathfrak{k}^* \arrow[d] \\
G \arrow[r, phantom, "\curvearrowright"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & (M,\omega) \arrow[rr, "\mu_G"] \arrow[rrd, bend left, "\mu_H"'] & & \mathfrak{g}^* \arrow[d] \\
H & & & \mathfrak{h}^*
\end{tikzcd}$$

If $1 \to H \xrightarrow{\trianglelefteq} G \longrightarrow K \longrightarrow 1$, then

(1) $K \curvearrowright M/\!\!/H \xrightarrow{\mu_K} \mathfrak{k}^*$

(2) $(M/\!\!/H)/\!\!/K = M/\!\!/G$

Eg. $T^N = \prod^N S^1 \curvearrowright \mathbb{C}^N \xrightarrow{\quad\mu\quad} \mathfrak{t}^* = \mathbb{R}^N$

$(e^{i\theta_j})_{j=1}^N \cdot (z_j)_{j=1}^N = (e^{i\theta_j} \cdot z_j)_{j=1}^N$

$$\mu\left((z_j)_{j=1}^N\right) = \left(\tfrac{1}{2}|z_j|^2 - c_j\right)_{j=1}^N$$

$\forall$ subtorus $T^{N-n} \leq T^N$ $\quad (\sim \mathbb{Z}^{N-n} \simeq \overset{\text{sublattice}}{\Lambda} \leq \mathbb{Z}^N)$

$\rightsquigarrow T^n \curvearrowright \mathbb{C}^N/\!\!/T^{N-n} =: X_\Lambda$ toric variety.

Eg. $U(r) \curvearrowright \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n) \xrightarrow{\quad\mu\quad} \underset{\{\text{Herm. matrices}\}}{\underset{\shortparallel}{\mathfrak{u}(r)^*}}$

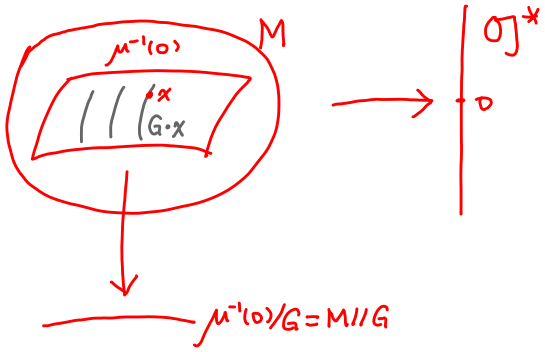
$$\mu(A_{r\times n}) = A \cdot A^*.$$

$$\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)/\!\!/U(r) \simeq Gr_{\mathbb{C}}(r,n).$$

Residual action, $U(n) \curvearrowright Gr_{\mathbb{C}}(r,n) \xhookrightarrow{\mu_{U(n)}} \mathfrak{u}(n)^*$

coadjoint orbit.

$G \curvearrowright (\underset{\substack{\rotatebox{90}{$\in$} \\ x}}{M}, \omega) \xrightarrow{\mu} \mathfrak{g}^*$



$d\mu(x): \; T_xM \longrightarrow \underbrace{T_{\mu(x)}\mathfrak{g}^*}_{\mathfrak{g}^*}$

Claim: $0 \to T_x(G\cdot x)^{\perp\omega} \longrightarrow T_xM \xrightarrow{d\mu(x)} \{\xi \in \mathfrak{g}^* : \langle \xi, \underbrace{\mathfrak{g}_x}_{\text{stabilizer}} \rangle = 0\} \to 0$

Cor: If $G \curvearrowright \mu^{-1}(0)$ free

$\Rightarrow \mathfrak{g}_x = 0 \Rightarrow d\mu(x)$ onto $\mathfrak{g}^*$ $\forall x \in \mu^{-1}(0)$

$\Rightarrow 0 \in \mathfrak{g}^*$ regular value of μ

$\Rightarrow$ submanifold $\mu^{-1}(0) \subset M$

Also $T_x(\mu^{-1}(0)) = \mathrm{Ker}(d\mu(x))$
$= T_x(G\cdot x)^{\perp\omega}$

Linearize: $G\cdot x \subset \mu^{-1}(0) \subset (M, \omega)$

$\rightsquigarrow \quad I \subset C \subset (V, \omega)$

$I^{\perp\omega} = C$

Namely, $C = I^{\perp\omega}$ is coisotropic

& $I = C^{\perp\omega}$ is isotropic

- $\omega(I, C) = 0 \Rightarrow \omega$ descends to $C/I \rightsquigarrow \omega_{red}$

- $(C/I, \omega_{red})$ Sympl. (i.e. ω_{red} non-degenerate)

$\iota_v\omega = 0$ on C/I

$\Rightarrow \omega(v, C) = 0 \Rightarrow v \in C^{\perp\omega} = I \Rightarrow v = 0 \in C/I$

$\rightsquigarrow$ Linear Symplectic reduction.

Cor. $(M//G, \omega_{red})$ non-deg. $\Rightarrow$ sympl.

Pf. of Claim: $T_xM \xrightarrow{d\mu(x)} \mathfrak{g}^*$

[reason: $d\mu^X = \iota_X \omega$]

$v \in \mathrm{Ker}(d\mu(x))$

$\Leftrightarrow \forall X \in \mathfrak{g}, \ \underbrace{d\mu(x)(v)(X)}_{\iota_X\omega(v) \, = \, \omega(v, X) \text{ at } x} = 0$

$\Leftrightarrow \quad v \in T(G \cdot x)^{\perp\omega}$

$\xi \in \mathrm{Im}(d\mu(x)) \subset \mathfrak{g}^*$

$\Leftrightarrow \quad \xi = d\mu(x)(v) \qquad \exists\, v \in T_xM$

$\Leftrightarrow \quad \xi(X) = d\mu(x)(v)(X) \quad \forall X \in \mathfrak{g}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$= \omega(v, \tilde{X}) \qquad \tilde{X} \in \mathrm{Vect}(M, \omega)$

$= 0$ if $\tilde{X}(x) = 0$

i.e. $X \in \mathfrak{g}_x$ stabilizer.

$\Rightarrow \quad \mathrm{Im}(d\mu(x)) \subseteq \{\xi \in \mathfrak{g}^* : \langle \xi, \mathfrak{g}_x \rangle = 0\}$

$\qquad\qquad\qquad\qquad\quad \uparrow$

dim. count $\Rightarrow$ $=$. QED.

A <u>variant</u>. $G \curvearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

$\forall$ coadj. orbit $\mathcal{O}_\xi \subset \mathfrak{g}^*$, (eg. $\mathcal{O}_\xi = \{0\}$)

$G \curvearrowright \mu^{-1}(\mathcal{O}_\xi)$ ($\because$ μ : G-inv.)

Similarly $M /\!\!/_\xi G := \mu^{-1}(\mathcal{O}_\xi)/G$ sympl.

Alternatively, $G \curvearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$

(combine w/ $G \curvearrowright (\mathcal{O}_\xi, \omega_\xi) \hookrightarrow \mathfrak{g}^*$)

$$\Rightarrow \quad G \curvearrowright (M \times \mathcal{O}_\xi, \omega - \omega_\xi) \xrightarrow{\tilde{\mu}} \mathfrak{g}^*$$
$$(x, \eta) \longmapsto \mu(x) - \eta$$

- $\mu^{-1}(\mathcal{O}_\xi) = \tilde{\mu}^{-1}(0)$

$$\Rightarrow M \times \mathcal{O}_\xi /\!/ G = \tilde{\mu}^{-1}(0)/G$$
$$= \mu^{-1}(\mathcal{O}_\xi)/G = M /\!/_\xi G$$
$$= \mu^{-1}(\xi)/G_\xi$$

$\swarrow$ stabilizer.

[reason: bundle:

$$\begin{tikzcd} \mu^{-1}(\xi) \arrow[d] & G_\xi \arrow[l, bend right, "\curvearrowleft" description, phantom] \arrow[d] \\ \mu^{-1}(\mathcal{O}_\xi) \arrow[d] & G \arrow[l, bend right, "\curvearrowleft" description, phantom] \arrow[d] \\ \mathcal{O}_\xi \arrow[r, phantom, "="] & G/G_\xi \end{tikzcd}$$

In general $(M,\omega) \supset C$ coisotropic (eg $\mu^{-1}(0)$)

$\Rightarrow \forall\, x \in C, \quad (T_xC)^{\perp\omega} \subseteq T_xC$

(($T_xC/(T_xC)^{\perp\omega}$, $\omega_{\text{red},x}$) sympl. v.s.)

⇝ (isotropic) distribution on C

$d\omega = 0 \implies$ integrable (i.e. foliation).

Pf: $X, Y \in \Gamma(C, (TC)^{\perp\omega})$, i.e. $\omega(X, TC) = 0 = \omega(Y, TC)$

$\overset{?}{\Rightarrow}$ $[X,Y] \in \Gamma($— " —$)$, i.e. $\omega([X,Y], TC) \overset{?}{=} 0$

$$0 \overset{d\omega=0}{=} d\omega(X, Y, Tc) = \omega([XY], Tc) \pm \cancel{\omega([X, Tc], Y)} \pm \cancel{\omega([Y, Tc], X)}$$
$$\pm Tc(\cancel{\underbrace{\omega(X,Y)}_{\text{isotropic.}}}) \pm X(\cancel{\omega(Y, Tc)}) \pm Y(\cancel{\omega(X, Tc)})$$

$\Rightarrow$ $(C/\sim, \omega_{\text{red}})$ Sympl. Reduction (if "smooth").

§ Existence of moments maps

$$G \curvearrowright (M, \omega) \qquad \underset{?}{\nexists}\ \mu$$

Recall: $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0 \Rightarrow \exists!\ \mu$

How about condition on M?

Prop: $[\omega] = c_1(L) \in H^2(M, \mathbb{Z})$

$$\begin{tikzcd} & L \arrow[d] \\ \text{cpt. } G \arrow[ur, bend left] \arrow[r, bend left] & M \end{tikzcd} \qquad \Longrightarrow \qquad \exists\ \mu$$

Pf. G cpt. $\Rightarrow \exists$ G-inv. conn. D_A on (L, h) s.t. $\omega = F_A$

$$\begin{tikzcd} & S^1 \arrow[d] \\ & P \arrow[d, "\pi"] \\ G \arrow[ur, bend left] \arrow[r, bend left] & M \end{tikzcd}$$

P ← unit sphere bundle of (L, h)

Conn. $\Rightarrow$ $\alpha \in \Omega^1(P)^{S^1}$

$\alpha|_{\text{fiber} \approx S^1} = d\theta$

curv. $\pi^* F_A = d\alpha$

$\forall\ X \in \mathfrak{g}$,

$\rightsquigarrow$ $\alpha(X) : P \longrightarrow \mathbb{R}$, S^1-inv.

descends $\underbrace{\alpha(X)}_{\mu^X} : M \longrightarrow \mathbb{R}$,

Eg. $\omega = d\alpha$ exact sympl. mfd

$\sim$ $L = M \times \mathbb{C}$

Theorem (Frenkel) $S^1 \curvearrowright (M, \omega, J)$ compact Kähler

$\exists$ fix point $\Longrightarrow$ $\exists\, \mu$

(i.e. $M^{S^1} \neq \phi$)

Proof : Use 2 facts :

(1) M^{2n} cpt Kähler,

$$\Longrightarrow H^1 \xrightarrow[\simeq]{\wedge\, \omega^{n-1}} H^{2n-1} \quad \text{isom}$$

(Special case of Hard Lefschetz theorem)

(2) (M, g) cpt Riemannian

$\mathcal{L}_X g = 0$, $\Delta \varphi = 0$ w/ $\varphi \in \Omega^1(M)$

$$\Longrightarrow \varphi(X) \equiv \text{const.}$$

(Prove later).

$$d(\iota_X \omega) = 0 \quad (\because \mathcal{L}_X \omega = 0)$$

$$\nexists\, \mu \Longleftrightarrow [\iota_X \omega] \overset{?}{\neq} 0 \quad \in H^1(M)$$

$$\overset{\text{by}(1)}{\Longleftrightarrow} \underbrace{[\iota_X \omega] \wedge [\omega^{n-1}]}_{\frac{1}{n}[\iota_X \omega^n]} \overset{?}{\neq} 0 \in H^{2n-1}(M)$$

$$\underset{\text{Hodge}}{\overset{\text{P.D.} +}{\Longleftrightarrow}} \int_M (\iota_X \omega^n) \wedge \varphi \overset{?}{\neq} 0 \qquad \forall\, \varphi \in \Omega^1(M),\ \Delta\varphi = 0$$

$$= - \int \underbrace{\underbrace{\underbrace{\iota_X(\varphi)}_{\varphi(X)}}_{\text{const.}\ (\text{by } (2))}}_{0 \quad (\because M^{S^1} \neq \phi)} \omega^n$$

Q.E.D.

To prove (2), first recall Riemannia Geometry:

$(M, g = g_{ij}(x)\, dx^i \otimes dx^j)$ cpt.

∃! Levi-Civita connection ∇ on TM,

$$\nabla = d + \Gamma^i_{jk} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \quad .$$

w/ $\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$

Curvature $\nabla^2 = Rm = R^i_{jkl}$

write $R_{ijkl} := g_{ip} R^p_{jkl} \quad = R_{klij}$

$$\left(R_{ijkl} + R_{iklj} + R_{iljk} = 0 \right)$$

Ricci curvature $R_{ij} = R_{ipjp} \quad = R_{ji}$

$Rm = \nabla^2 \Leftrightarrow \forall\, X = X^i \frac{\partial}{\partial x^i}\,,\ X^i{}_{,kl} - X^i{}_{,lk} = X^j R^i_{jkl}$

$\Leftrightarrow \forall\, \varphi = \varphi_i dx^i\,,\ \varphi_{i,kl} - \varphi_{i,lk} = -\varphi_j R^j_{ikl}$

Lemma: (i) $\mathcal{L}_X g = 0 \Rightarrow \nabla^*\nabla X = Rc(X) \in \Gamma(TM)$

(ii) $\Delta \varphi = 0 \Rightarrow \nabla^*\nabla \varphi = -Rc(\varphi) \in \Omega^1(M)$

Cor: $Rc < 0 \Rightarrow$ ∄ Killing vector field,

i.e. $Aut(M, g)$ discrete.

$Rc > 0 \Rightarrow$ ∄ harmonic 1-form,

i.e. $H^1(M, \mathbb{R}) = 0$

Pf. of Cor: $\int \langle \nabla^*\nabla X, X \rangle = \int |\nabla X|^2 \geq 0$

$= \int \langle Rc(X), X \rangle \leq -\int |X|^2 \leq 0$

$\Rightarrow |X| \equiv 0$ i.e. $X = 0$

Similar for φ . QED.

Pf. of lemma: (i) $\mathcal{L}_X g = \mathcal{L}_{X^i\partial_i}(g_{jk}\, dx^j \otimes dx^k)$

$$= X^i g_{jk,i} + g_{jk} \underbrace{\underbrace{d(X(x^j))}_{dX^j}}_{X^j{}_{,i}\,dx^i} dx^k + g_{jk}\, dx^j \underbrace{\underbrace{d(X(x^k))}_{dX^k}}_{X^k{}_{,i}\,dx^i} \quad (\because \mathcal{L}d = d\mathcal{L})$$

$$\overset{\text{In normal coord.}}{\underset{g=\delta+O(|x|^2)}{=\!=\!=}} (X^k{}_{,i} + X^i{}_{,k})\, dx^i \otimes dx^k$$

$$\nabla^*\nabla X = X^j{}_{,ii} \overset{\text{by above}}{\underset{(\because \mathcal{L}_X g = 0)}{=\!=\!=}} - X^i{}_{,ji}$$

$$= -(\underbrace{X^i{}_{,ij}}_{(\cancel{\mathrm{div} X})_{,j}} + \underbrace{R^i_{kji} X^k}_{-Rc(X)})$$

$(\because X^i{}_{,j} + X^j{}_{,i} = 0)$

(ii) $$\Delta\varphi = (d^*d + dd^*)\varphi$$

$$= d^*(\varphi_{i,j} - \varphi_{j,i}) + d(\varphi_{j,j})$$

$$= \underbrace{\varphi_{i,jj}}_{\nabla^*\nabla\varphi} \underbrace{- \varphi_{j,ij} + \varphi_{j,ji}}_{- \varphi_k R^k_{jji}}$$

$$= \nabla^*\nabla\varphi + Rc(\varphi).$$

Pf. of $[\mathcal{L}_X g = 0,\ \Delta\varphi = 0 \in \Omega^1 \overset{?}{\Rightarrow} \varphi(X) \equiv \text{const.}]$

$$\Delta(\varphi(X)) = (\varphi_i X^i)_{,jj} = \underbrace{\varphi_{i,jj}}_{(-R^k_i\varphi_k)} X^i + 2\varphi_{i,j} X^i{}_{,j} + \varphi_i \underbrace{X^i{}_{,jj}}_{(R^i_k X^k)}$$

$$(-R^k_i\varphi_k)X^i \underset{\text{cancel.}}{\longleftrightarrow} \varphi_i\cdot(R^i_k X^k)$$

$$= 2\varphi_{i,j} X^i{}_{,j}$$

$$= 0 \qquad (\because \varphi_{i,j} = \varphi_{j,i}\ ;\ X^i{}_{,j} = -X^j{}_{,i})$$

$$\xRightarrow[\text{max. pr.}]{M \text{ cpt.}} \varphi(X) \equiv \text{const.}$$

QED.

Theorem (McDuff) $S^1 \curvearrowright (M^4, \omega)$ cpt.

$\exists$ fix point $\Longrightarrow$ $\exists \, \mu$

Need some general facts:

(1) cpt. $G \curvearrowright (M, \omega)$

$\Rightarrow$ $G \curvearrowright (M, \omega, g, J)$

$\exists$ G-inv. compat. g & J

(Note: J is only an alm. cpx. str.)

i.e. $\exists$ G-inv. almost Kähler structure.

Reason: Averaging ⇝ G-inv. g_0

define $A \in \mathrm{End}(T_M)$ by $\omega(u, Av) = g_0(u, v)$.

$\Rightarrow$ A is non-sing, skew-symm., G-inv.

$\Rightarrow$ $B = -A^2 > 0$, G-inv.

$$g(u,v) \triangleq g_0(u, B^{-\frac{1}{2}} v) = \omega(u, \underbrace{AB^{-\frac{1}{2}}}_{J} v)$$

$$J^2 = -id \quad (\because B = -A^2)$$

In fact, $\mathcal{J} := \{\text{compatible } J\}$ is contractible.

(important for constructing Sympl. inv., eg. GW-inv.)

reason: $$\mathcal{J} = \Gamma(M, \mathcal{F} \times_{Sp(2n,\mathbb{R})} \underbrace{\{\text{compat. } J \text{ on } (\mathbb{R}^{2n}, \omega_{std})\}}_{\sim \underbrace{Sp(2n,\mathbb{R})/U(n)}_{\text{contractible.}}})$$

where $Sp(2n, \mathbb{R}) \longrightarrow \mathcal{F} \longrightarrow M$

is principal symplectic frame bundle.

recall: frame on (V, ω) is $(V, \omega) \longrightarrow (\mathbb{R}^{2n}, \omega_{std})$.

(2) $G \curvearrowright (M, \omega, g, J)$ alm. Kähler.

$\Longrightarrow$ $M^G \subset M$ is J-holo. submfd.

($\Longrightarrow$ $M^G \subset M$ is sympl. submfd.)

$\Longrightarrow$ $N_{M^G/M}$ Cpx. VB/M^G, G-bdl.

(reason: $[X, JY] = \mathcal{L}_X(JY) = \cancel{(\mathcal{L}_X J)}Y + J\mathcal{L}_X Y = J[X,Y]$)

- $\omega(v, Jv) = g(Jv, Jv) > 0$

So cpx. subsp. $\Rightarrow$ sympl. subsp.

(3) $G \curvearrowright (M, \omega)$

$\Rightarrow$ $G \curvearrowright (M, \omega'/\mathbb{Q})$ $\exists$ sympl. $\omega' \overset{\text{close}}{\sim} \omega$

(reason: $[\omega] \in H^2(M, \mathbb{R})^G = H^2(M, \mathbb{Q})^G \otimes_{\mathbb{Q}} \mathbb{R}$)

After scaling, $G \curvearrowright (M, \omega''/\mathbb{Z})$.

(4) $[M, S^1] \cong H^1(M, \mathbb{Z})$

$$f \longleftrightarrow [f^*(d\theta)]$$

(reason: $S^1 = K(\mathbb{Z}, 1)$)

(5) $S^1 \curvearrowright (M, \omega/\mathbb{Z})$

$\Rightarrow \exists\ \hat{\mu} : M \longrightarrow \mathbb{R}\underbrace{/\mathbb{Z}}$

s.t. $\iota_X \omega = -d\hat{\mu}$ $\quad$ ($X = \frac{\partial}{\partial \theta}$)

(reason: $[\iota_X \omega] \in H^1(M, \mathbb{Z}) = [M, S^1]$)

Lemma: $S^1 \curvearrowright (M, \omega/\mathbb{Z})$

$$\nexists\ \mu \implies \operatorname{codim} M^{S^1} \geqslant 4$$

Pf: $\hat{\mu} : M \longrightarrow \mathbb{R}/\mathbb{Z}$

$p \in \operatorname{Crit}(\hat{\mu}) = M^{S^1}$ $\quad (\because -d\hat{\mu} = \iota_X \omega)$

p loc. min./max.

$\xRightarrow{\text{(Exercise)}}$ $\exists$ lift
$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[rd, "\hat{\mu}"'] & \mathbb{R} \arrow[d] \\ & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$
$\Rightarrow \exists \mu \ (-\!\!\ast\!\!-)$

So $\operatorname{codim} \operatorname{Crit}(\hat{\mu}) = \text{index}\, p + \text{coindex}\, p$

$$\geqslant \overset{\nearrow}{2} + 2 = 4 \quad \text{QED.}$$

$(N_{M^G/M}$: cpx G-bdl. $\Rightarrow$ (co-) index $\in 2\mathbb{Z}$)

Proof of McDuff theorem:

$$S^1 \curvearrowright (M^4, \omega/\mathbb{Z}) \xrightarrow{\hat{\mu}} \mathbb{R}/\mathbb{Z}$$

$\nexists\ \mu \xRightarrow{\text{lemma}} 0 \in M^{S^1}$ discrete set

$\xRightarrow[\text{(for } \omega\text{, NOT } J)]{S^1\text{-Darboux}}$ locally: $S^1 \curvearrowright \mathbb{C}^2$, $\quad p, q \geqslant 1$

$$e^{i\theta} \cdot (z_1, z_2) = (\underbrace{e^{ip\theta} z_1}_{\text{coindex } 2},\ \underbrace{e^{-iq\theta} z_2}_{\text{index } 2})$$

$$\hat{\mu} = p|z_1|^2 - q|z_2|^2$$

$\hat{\mu}^{-1}(t)$ $\quad$ Seifert fibration if $t \neq 0$

$\downarrow$

$\hat{\mu}^{-1}(t)/S^1 \leftarrow$ orbifold

$$\mathbb{Q} \ni \chi\left(\begin{array}{c}\hat{\mu}^{-1}(t)\\ \downarrow \\ \hat{\mu}^{-1}(t)/S^1\end{array}\right) = \chi\left(\begin{array}{c}\hat{\mu}^{-1}(-t)\\ \downarrow \\ \hat{\mu}^{-1}(-t)/S^1\end{array}\right) - \frac{1}{pq} \quad \text{for } t > 0 \qquad (*)$$

but $\hat{\mu} \in \mathbb{R}/\mathbb{Z}$

$\chi_{t=\frac{1}{2}} = \chi_{t=-\frac{1}{2}}$ $\longrightarrow (-\!\!\ast\!\!-)$

$\Rightarrow M^{S^1} = \emptyset$ $\quad$ QED.

§ $M/G^{\mathbb{C}}$ vs $M/\!/G$

GIT quotient vs Symplectic quotient
(Geometric Invariant Theory)

e.g. $\underbrace{\mathbb{C}^2/\mathbb{C}^\times}_{\mathbb{C}^2\setminus 0/\mathbb{C}^\times} = \underbrace{\mathbb{C}^2/\!/S^1}_{S^3/S^1} = \mathbb{CP}^1$

- $G \curvearrowright (M,\omega,J)$ Kähler

i.e. G acts by holomorphic isometries $(\because \omega + J \Rightarrow g)$

$\Rightarrow$ (i) Can assume G compact.
($\because$ $\mathrm{Aut}(M,g)$ compact.)

e.g. $\mathrm{Aut}(\mathbb{CP}^n, g_{FS}) \cong \mathbb{P}U(n+1)$

$\mathrm{Aut}(\mathbb{CP}^n, J) \cong \mathbb{P}GL(n+1,\mathbb{C})$

$\cong$ Complexification of $\mathbb{P}U(n+1)$.

(ii) $G^{\mathbb{C}} \curvearrowright (M,J)$
($\because$ J integrable).

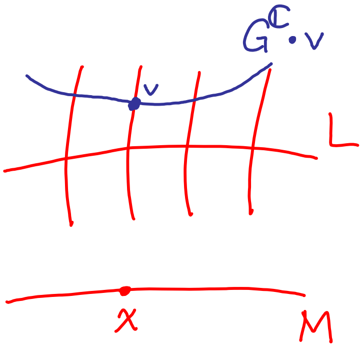
- $G^{\mathbb{C}} \overset{\rho}{\curvearrowright} (M,J)$ vs $G \overset{\rho}{\curvearrowright} (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$

$\Downarrow$ $\Downarrow$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "d\rho"] & T_xM \arrow[dd, bend left, "J"] \\
\circlearrowleft & \\
i\mathfrak{g} \arrow[r, "Jd\rho"'] & T_xM
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathfrak{g} \arrow[r, "d\rho"] & T_xM \arrow[dddd, bend left, "\lrcorner\omega"] \\
& \circlearrowleft & \\
T_xM \arrow[rr, "d\mu"] & & \mathfrak{g}^* \\
& \parallel & \\
& \mathfrak{g} \arrow[r] & T_x^*M
\end{tikzcd}$$

Setting: Assume $G \curvearrowright (M, \underbrace{\omega/\mathbb{Z}}, J)$ Kähler

$$\Rightarrow \quad \begin{tikzcd} & (L, h) \arrow[d, "\pi", "\mathbb{C}"'] & D_A h = 0 \\ G \arrow[ur, bend left] \arrow[r, bend left] & (M, \omega, J) & \text{Kähler} \\ & F_A = \omega. & \end{tikzcd}$$



Fix any $v \in L_x \setminus 0$, define

$$H : G^{\mathbb{C}}/G \longrightarrow \mathbb{R}$$

$$H(g) = \log |g \cdot v|_h$$

(well def^{d}, ∵ G preserve h ⇒ (i) descend to $/G$; (ii) $|g \cdot v| \neq 0$)

- $T^*G \simeq G \times \mathfrak{g}^* \overset{K}{\simeq} G \times \mathfrak{g} \xrightarrow[\text{diffeo.}]{\sim} G^{\mathbb{C}}$

$$(g, X) \longmapsto g \cdot e^{iX}$$

In particular, $\mathfrak{g} \xrightarrow{\sim} G^{\mathbb{C}}/G$

$$X \longmapsto e^{iX}$$

Write $H(e^{itX}) =: H_X(t) : \mathbb{R} \longrightarrow \mathbb{R}$

(namely, restrict^{n} of H to 1-parameter subgp. of $G^{\mathbb{C}}$)

Prop: (i) $g \in \mathrm{Crit}(H) \Longleftrightarrow g \cdot x \in \mu^{-1}(0)$

(ii) $H_X'(t) = -2 \mu^X(t)$

$$H_X''(t) = 2 \, | d\rho_{x_t}(X) |^2_{T_xM} \geqslant 0$$

$$x_t := e^{itX} \cdot x.$$

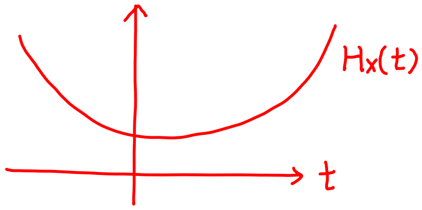
Cor. $G^{\mathbb{C}} \cdot x \cap \mu^{-1}(0) \neq \phi$

i.e. $\exists\, g \in G^{\mathbb{C}}$ s.t. $\mu(g \cdot x) = 0$

$\iff \forall\, X \in \mathfrak{g} \setminus \mathfrak{g}_x$

$H_X(t)$ has a unique min.

$\iff \forall\, X \in \mathfrak{g} \setminus \mathfrak{g}_x$



$\lim_{t \to \infty} H_X'(t) > 0$

$\overset{\Delta}{\iff}$ $x \in M$ is polystable

Cor. $M^{s.s.}/G^{\mathbb{C}} \underset{\text{homeo.}}{\simeq} \mu^{-1}(0)/G$

i.e. every polystable $G^{\mathbb{C}}$-orbit contains a unique G-orbit in $\mu^{-1}(0)$.

i.e. $M/G^{\mathbb{C}} \simeq M/\!/G$.

Eg. $L = \mathbb{C}^{n+1} \times \mathbb{C}$, $e^{i\theta} \cdot ((z_j)_{j=0}^{n}, w) = ((e^{i\theta} z_j)_{j=0}^{n}, e^{i\theta} w)$

$$\begin{tikzcd} & L \arrow[d] \\ S^1 \arrow[ur, bend left] \arrow[r, bend left] & (\mathbb{C}^{n+1}, \omega, J) \arrow[r, "\mu = \frac{1}{2}(r^2 - 1)"] & \mathbb{R} \end{tikzcd}$$

$\mu^{-1}(0) = \underset{\text{unit sphere}}{S^{2n+1}} \subset \mathbb{C}^{n+1}$

$\vec{z} \in \mathbb{C}^{n+1} \Rightarrow G^{\mathbb{C}} \cdot \vec{z} = \{ c\vec{z} \mid c \in \mathbb{C}^{\times} \}$

$\vec{z} \neq 0 \Rightarrow$ take $|c| = \frac{1}{|\vec{z}|}$, then $c\vec{z} \in S^{2n+1}$

$G^{\mathbb{C}} \cdot \vec{z} \cap S^{2n+1} = G \cdot (c\vec{z})$

Proof of Prop. [(ii) $\Rightarrow$ (i) ✓]

Choose local <u>holo.</u> trivializat$^{\underline{n}}$ $L|_U \cong U \times \mathbb{C}$,

then metric on $L|_U \sim$ $h : U \longrightarrow \frac{GL(1,\mathbb{C})}{U(1)} \simeq i\,\underline{u}(1) \cong \mathbb{R}$.

$$\omega = d\alpha \quad \text{w/} \quad \alpha = h^{-1}\partial h \qquad (\text{forget } i\,)$$

$$\Rightarrow \mu^X = \alpha(X) = X(\log h)$$

$$H_X'(0) = \frac{d}{dt}\Big|_{t=0} \log \underbrace{|e^{itX}\cdot v|_{L_{x_t}}}_{h(x_t)\underbrace{|e^{itX}\cdot v|_{\mathbb{C}}}_{\text{const. in } t}} \qquad \left(\begin{array}{l}\text{indep. of choice} \\ \text{of } v \in L_x \setminus 0\end{array}\right)$$

$(\because S^1 \curvearrowright \mathbb{C} = L_x \text{ rotates})$

$$= X(\log h) = \mu^X(x) \qquad \checkmark \ (\text{up to } -2\,)$$

$$H_X'(t) = -2\,\mu^X(x_t) \quad \text{w/} \quad x_t = e^{itX}\cdot x$$

Along $e^{itX}\cdot x$, $\frac{d}{dt} \longleftrightarrow J(\tilde{X})$

$$\Rightarrow H_X''(t) = -2\;J\tilde{X}(\mu^X) \quad = -2\;d\mu^X(J\tilde{X})$$

$$= -2\;\omega(\tilde{X}, J\tilde{X})$$

$$= 2\;g(\tilde{X}, \tilde{X}) = 2|\tilde{X}|^2$$

QED.

§ Obstructions to stability.

Always assume $\underset{\text{cpt.}}{G} \overset{\rho}{\curvearrowright} \underset{\text{cpt. Kähler}}{(M, \omega, J)} \xrightarrow{\mu} \mathfrak{g}^*$

Futaki invariant. Given $x \in M$, $X \in \mathfrak{g}_x^{\mathbb{C}}$,

consider $F(X) : G^{\mathbb{C}} \longrightarrow \mathbb{C}$

$$F(X)(g) = \langle \underbrace{\mu(g\cdot x)}_{\mathfrak{g}^*}, \underbrace{\mathrm{Ad}_g(X)}_{\mathfrak{g}^{\mathbb{C}}} \rangle$$

Prop. $F(X)(g) \equiv \text{const.} = \langle \mu(x), X \rangle$

Cor: $F(X) = \langle \mu(x), X \rangle : \mathfrak{g}_x^{\mathbb{C}} \longrightarrow \mathbb{C}$

$\Rightarrow$ (i) indep. of $x \in G^{\mathbb{C}}$-orbit

(ii) F Lie alg. homomorphism.

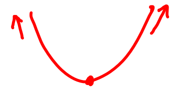
(iii) $F \equiv 0$ if $G^{\mathbb{C}}\cdot x \cap \mu^{-1}(0) \neq \emptyset$

Prop. $x_0 \in \overline{G^{\mathbb{C}}\cdot x} \subset M$ w/ $\mathfrak{g}_{x_0}^{\mathbb{C}} \neq 0 = \mathfrak{g}_x^{\mathbb{C}}$

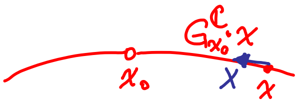
$\Rightarrow$ $G^{\mathbb{C}}\cdot x \cap \mu^{-1}(0) = \emptyset$ (i.e. unstable).

Pf. If NOT, then ∃ min. for convex fu. H_x $\Rightarrow$

$\forall X \in \mathfrak{g}_x \setminus 0$

$$\lim_{t\to\infty} H'_X(t) \underset{\sim\sim\sim}{>0}$$ $(\because$  $)$

$$\overset{\text{Prop.}}{=} 2 \underbrace{\langle \mu(x_0), X \rangle_{\mathfrak{g}}}_{\iota_{\underbrace{\tilde{X}(x_0)}_{\underset{\sim\sim}{0}}} \omega}$$



if choose $X \in \mathfrak{g}_{x_0} \setminus \mathfrak{g}_x$

contradiction.

- $(G^{\mathbb{C}} \cdot x) \cap \mu^{-1}(0) \neq \phi$

  $\Rightarrow \quad \mathfrak{g}_x^{\mathbb{C}}$ reductive.

More generally, we have the following structure result.

Theorem. $x \in \mathrm{Crit}(|\mu|^2)$ extremal point

(Pf. omitted).

$$\mathrm{ad}_{i\mu(x)} : \mathfrak{g}_x^{\mathbb{C}} \longrightarrow \mathfrak{g}_x^{\mathbb{C}}$$

eigenspace $\mathfrak{h}_\lambda := \{ X \in \mathfrak{g}_x^{\mathbb{C}} : [i\mu(x), X] = \lambda X \}$

$$\Rightarrow \quad \mathfrak{g}_x^{\mathbb{C}} = \mathfrak{h}_0 \bigoplus_{\lambda > 0} \mathfrak{h}_\lambda \qquad \text{s.t.}$$

(i) $\mathfrak{h}_0$ = reductive part of $\mathfrak{g}_x^{\mathbb{C}}$,

(ii) $[\mathfrak{h}_{\lambda_1}, \mathfrak{h}_{\lambda_2}] \subset \mathfrak{h}_{\lambda_1 + \lambda_2}$.

(iii) $\mu(x) \in \mathrm{Center}(\mathfrak{h}_0)$.

Cor. 1° $\mu(x) = 0 \Rightarrow \mathfrak{g}_x^{\mathbb{C}}$ reductive

2° $\left.\begin{array}{l} x \text{ extremal} \\ \langle \mu(x), \mathfrak{h}_0 \rangle_{\mathfrak{g}^{\mathbb{C}}} = 0 \end{array}\right\} \Rightarrow \mu(x) = 0$

3° $\left.\begin{array}{l} x \text{ extremal} \\ \mu(x) \neq 0 \end{array}\right\} \Rightarrow \exists\, \mathbb{R} \subset \mathfrak{g}_x^{\mathbb{C}}$.

Lemma: $\nabla |\mu|^2(x) = 2J\,(d\rho_x\,(\mu(x))) \in T_xM$.

$$\left(\mu(x) \in \mathfrak{g}^* \simeq \mathfrak{g} \xrightarrow{d\rho_x} T_xM \xrightarrow{J} T_xM \right)$$

Pf: $\nabla |\mu|^2_{\mathfrak{g}}(x) = 2 \langle \mu(x), \nabla \mu(x) \rangle_{\mathfrak{g}}$

$$g(\nabla|\mu|^2, \tilde{Y})_{T_xM} = 2 \langle \mu(x), \underbrace{\nabla_{\tilde{Y}} \mu(x)}_{\iota_{\tilde{Y}}\omega} \rangle_{\mathfrak{g}}$$

$$= 2\,\omega(d\rho_x(\mu(x)), \tilde{Y})$$

$$= 2g(J(\text{—''—}), \tilde{Y})$$

QED.

Exercise: Given symplectic vector space (V, ω) and a compatible J ($\rightsquigarrow g$).

i.e. $(V, \omega, J, g) \simeq (\mathbb{C}^n, \omega_{std}, J_{std}, g_{std})$

- $U(n) \leqslant Sp(2n, \mathbb{R}) \overset{\text{both}}{\underset{\text{transitive}}{\curvearrowright}} \{L \subset V : \text{Lagr}\} \triangleq \mathcal{L}(n)$

w/ isotropy gp. $O(n) \leqslant U(n)$ & $\left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\} \leqslant Sp(2n, \mathbb{R})$

$\Longrightarrow \mathcal{L}(n) = \{\text{Lagr}\} = U(n)/O(n) = Sp(2n, \mathbb{R}) / \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$

- How about 2 Lagr. $L_1, L_2 \subset V$?

(1) $\exists\ g \in Sp(2n, \mathbb{R})$ s.t. $(L_1', L_2') = (gL_1, gL_2)$

$\Longleftrightarrow \quad \dim L_1' \cap L_2' = \dim L_1 \cap L_2$

(2) $\exists\ g \in U(n)$ s.t. $(L_1', L_2') = (gL_1, gL_2)$

$\Longleftrightarrow \quad P(L_1', L_2') = P(L_1, L_2)$

(Here $P(L_1, L_2) =$ char. poly. of AA^t

$A = (\underset{\substack{\uparrow \\ \text{Hermitian metric}}}{h}(v_j, u_i))$ $\quad$ u_i's o.n. basis for L_1
$\quad$ v_j's o.n. basis for L_2

(3) $\exists$ o.n. bases s.t. $v_j = e^{i\lambda_j} u_j \quad \forall j$.

We have $e^{2i\lambda_j}$'s as roots of $P(L_1, L_2)$.

In particular, $\exists\ \varphi \in U(n)$, $\varphi(L_1) = L_2$

(i.e. $\varphi(u_j) = v_j$ & $\varphi(Ju_j) = Jv_j$)

(Lagr. $\longleftrightarrow$ anti-holo. involution.
$L = V^{\sigma_L}$ $\quad$ $\sigma_L|_L = 1$ & $\sigma_L|_{JL} = -1$)

$\varphi^2 = \sigma_{L_2} \circ \sigma_{L_1}$

(4) $$\frac{\mathcal{L}(n)\times\mathcal{L}(n)}{U(n)} \xrightarrow{\textcircled{H}} T/W$$

$$(L_1, L_2) \longmapsto (e^{2i\lambda_1}, \dots, e^{2i\lambda_n})$$

is a homeo.

- How about 3 Lagr. $L_1, L_2, L_3 \subset V$?

(5) $\exists\, g \in Sp(2n,\mathbb{R})$ s.t. $L_j' = gL_j \ \forall j = 1,2,3$

$\iff$ Same $\underbrace{\dim L_1\cap L_2\cap L_3}_{n_0}$, $\underbrace{\dim L_j\cap L_k}_{n_{jk}}$, τ.

In particular, $Sp(2n,\mathbb{R}) \curvearrowright \prod^3 \mathcal{L}(n)$ has finite number of orbits.

Here $\tau := \text{Signature}\left(q : L_1\oplus L_2\oplus L_3 \xrightarrow{\text{quad. form}} \mathbb{R}\right)$

$$q(x_1,x_2,x_3) := \omega(x_1,x_2)+\omega(x_2,x_3)+\omega(x_3,x_1)$$

Properties: $n_{12}+n_{23}+n_{31} \leqslant n + 2n_0$

$\tau \equiv n - (\text{— '' —})$ (mod 2)

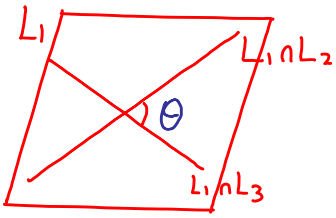
$|\tau| \leqslant n - (\text{— '' —}) + 2n_0$

$-\tau + 3n - (\text{— '' —}) = 2\delta/\pi$

where $\delta = \mathrm{Tr}\,\textcircled{H}(L_1,L_2) + \mathrm{Tr}\,\textcircled{H}(L_2,L_3) + \mathrm{Tr}\,\textcircled{H}(L_3,L_1)$.

(6) $\exists\, g \in U(n)$ s.t. $L_j' = gL_j \ \forall j = 1,2,3$

$\overset{n=2}{\iff}$ same $\textcircled{H}(L_1,L_2)$, & θ :
$\textcircled{H}(L_1,L_3)$



§ Contact Geometry

(Odd dim. analog of symplectic geometry).

eg. $\underset{S^1 \times \mathbb{R}_t}{\overset{\mathbb{R}^2 \setminus 0}{\shortparallel}}$, $\omega = dx \wedge dy = r\,dr \wedge d\theta$

$$= d(\underbrace{r^2}_{e^t} \underbrace{\tfrac{1}{2} d\theta}_{\alpha})$$

$$\Rightarrow \mathcal{L}_{\frac{\partial}{\partial t}} \omega = \omega.$$

Same for $\mathbb{R}^{2n+2} \setminus 0 = S^{2n+1} \times \mathbb{R}_{t = \log r^2}$

Ex. $Y^{2n+1} \times \mathbb{R}_t = M^{2n+2}$, ω sympl.

$$\mathcal{L}_{\frac{\partial}{\partial t}} \omega = \omega \iff \omega = d(e^t \alpha)$$

$$\alpha \in \Omega^1(Y)$$

ω : non-deg. $\iff \alpha \wedge (d\alpha)^n \neq 0$

$\omega^{n+1} \neq 0$

Def. $\alpha \in \Omega^1(Y^{2n+1})$ contact form

if $\alpha \wedge (d\alpha)^n \neq 0$ (non-vanishing).

($\Rightarrow$ $Y \times \mathbb{R}_t$, $\omega = d(e^t \alpha)$ sympl).

Darboux / Moser ✓. eg. locally $\alpha = \Sigma x^j dy_j + dz$.

Def. Reeb vector field $R \in \Gamma(Y, T_Y)$:

$$\iota_R d\alpha = 0 \quad \& \quad \iota_R \alpha = 1$$

Ex. J compat. alm. cpx. str./ $Y\times\mathbb{R}_t$ s.t. $\mathcal{L}_{\frac{\partial}{\partial t}} J = 0$ (translation inv.).

$\gamma\times\mathbb{R} \subset Y\times\mathbb{R}$ is J-holo.

iff γ is a Reeb orbit.

Weinstein Conj. / Taubes Theorem.

(Y^3, α) contact

$\Longrightarrow$ $\exists$ closed Reeb orbit.

Ex. $K^n\times\mathbb{R} \subseteq Y^{2n+1}\times\mathbb{R}$ is Lagrangian

$\Longleftrightarrow$ $d\alpha|_K = 0 \in \Omega^2(K)$

called Legendrian submanifold.

($\rightsquigarrow$ Reeb chord w/ $\partial \subset$ Legendrains).

$\sim$ Legendrian contact homology.

- $\alpha \in \Omega^1(Y^{2n+1}) \rightsquigarrow 0 \to \underbrace{\mathrm{Ker}\,\alpha}_{\substack{H^{2n}\\ \text{contact hyperplane bdl.}}} \to TY \to \mathcal{L}^1 \to 0$

$\alpha\wedge(d\alpha)^n \neq 0 \Longleftrightarrow d\alpha|_H$ non-degenerate.

$\Longrightarrow TY = \mathrm{Ker}\,\alpha \oplus \mathrm{Ker}(d\alpha)$

Locally, α is uniquely determined by H, up to scaling by nonvanishing functions. Such H is defined a contact structure.

- $\exists\, \alpha$ globally $\Longleftrightarrow TY/H = \mathcal{L}^1$ trivial line bdl/ Y